Sub-$n$-normal Operators

Il Bong Jung, Eungil Ko and Carl Pearcy

Abstract. In this paper we introduce the class of sub-$n$-normal operators. By definition, such an operator is the restriction to an invariant subspace of an $n$-normal operator, and thus the sub-$n$-normal operators form a larger class than the subnormal operators. We obtain some modest structure theorems and contrast sub-$n$-normal operators with sub-Jordan operators. Finally we show that a sub-$n$-normal operator with rich spectrum has a nontrivial invariant subspace.

Mathematics Subject Classification (2000). Primary 47B20, Secondary 47B15.

Keywords. Sub-$n$-normal operator, subnormal operator, sub-Jordan operator.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. In this paper we initiate a study of a class of operators — the union (over the set $\mathbb{N}$ of positive integers) of the classes of sub-$n$-normal operators — to be defined below. As will be seen, this union contains the much-studied classes of normal and subnormal operators, as well as the union of the somewhat less studied classes of $n$-normal operators, and thus may be of some interest, especially since we are only able to solve the invariant subspace problem for these operators in a special case (cf. Theorem 5.1 below).

We begin by recalling a definition of the class of $n$-normal operators in $\mathcal{L}(\mathcal{H})$. If $n \in \mathbb{N}$ and $\mathcal{K}$ is a Hilbert space, we write $\mathcal{K}^{(n)}$ for the direct sum of $n$ copies of $\mathcal{K}$. Fix now an $n \in \mathbb{N}$. An operator $T$ in $\mathcal{L}(\mathcal{H})$ is called an $n$-normal operator if there exist a complex Hilbert space $\mathcal{K}$ and a unitary isomorphism $\varphi$ of $\mathcal{H}$ onto $\mathcal{K}^{(n)}$ such that $\varphi T \varphi^{-1}$ is an $n \times n$ operator matrix $(N_{ij})$ acting on $\mathcal{K}^{(n)}$ in the usual fashion, where the entries $N_{ij}$, $i, j = 1, 2, ..., n$, are mutually commuting normal operators in $\mathcal{L}(\mathcal{K})$. The theory of $n$-normal operators is well-developed. It was begun in [3] and continued in [18], [19], [9], [10], and [15], to mention but some pertinent papers. In particular, one knows a complete set of unitary invariants...
for an \( n \)-normal operator [18], and that every nonscalar \( n \)-normal operator has a nontrivial hyperinvariant subspace [15].

Obviously the classes of normal and 1-normal operators in \( \mathcal{L}(\mathcal{H}) \) coincide, and there is some overlap between the classes of \( m \)-normal and \( n \)-normal operators. (For example, every normal operator of uniform multiplicity \( n \) is also an \( n \)-normal operator, etc.)

**Definition 1.1.** An operator \( T \in \mathcal{L}(\mathcal{H}) \) is a sub-\( n \)-normal operator (for some fixed \( n \in \mathbb{N} \)) if there exist an \( n \)-normal operator \( N \) and an invariant subspace \( M \) for \( N \) (necessarily of dimension \( \aleph_0 \)) such that \( T \) is unitarily equivalent to the restriction \( N|_M \). Note that by definition all sub-\( n \)-normal operators act on an infinite dimensional Hilbert space. (This avoids uninteresting contexts.)

Thus, up to unitary equivalence, the most general sub-\( n \)-normal operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) has an \( n \)-normal extension \( N = (N_{ij})_{i,j=1}^n \) acting on some Hilbert space \( \mathcal{K}^{(n)} \supset \mathcal{H} \), (i.e., \( NH \subset H \) and \( N|_H = T \)). Obviously all \( n \)-normal operators in \( \mathcal{L}(\mathcal{H}) \) are sub-\( n \)-normal operators. Moreover, the classes of subnormal and sub-1-normal operators in \( \mathcal{L}(\mathcal{H}) \) coincide, and, once again, there is overlap between the classes of sub-\( n \)-normal and sub-\( m \)-normal operators. (But these overlaps will not concern us.) We are interested in pursuing questions such as 1) which operators are (and which are not) sub-\( n \)-normal operators?, 2) what can be said about the structure of sub-\( n \)-normal operators?, and 3) does every sub-\( n \)-normal operator have a nontrivial invariant subspace?

### 2. Universality

Perhaps the most obvious question should be dealt with first. Is every operator in \( \mathcal{L}(\mathcal{H}) \) a sub-\( n \)-normal operator for some \( n \in \mathbb{N} \)? The answer is “no”, but it is not quite so obvious as one might suppose.

**Lemma 2.1.** Suppose \( n \in \mathbb{N}, \mathcal{K} \) is a complex Hilbert space, and \( A = (A_{ij})_{i,j=1}^n \) is an \( n \times n \) operator matrix acting on \( \mathcal{K}^{(n)} \) with the properties that \( A_{ij} = 0 \) whenever \( i > j \) (i.e., the matrix \( A \) is in upper triangular form) and for all \( i = 1, \ldots, n \), \( \ker A_{ii} = \ker A^n_2 \). Then \( \ker A^{n+1} = \ker A^n \) (and hence \( \ker A^{n+k} = \ker A^n \) for all \( k \in \mathbb{N} \)).

**Proof.** The proof is by induction on \( n \), and for \( n = 1 \) the result follows from the hypothesis. Suppose then that \( m \in \mathbb{N} \backslash \{1\} \) and that the result has been established for all \( (m-1) \times (m-1) \) operator matrices with the appropriate properties. Suppose, moreover, that we are given an \( m \times m \) matrix \( A = (A_{ij})_{i,j=1}^m \) acting on \( \mathcal{K}^{(m)} \) and satisfying the hypotheses of the lemma. We write \( \mathcal{K}^{(m)} \) as \( \mathcal{K}^{(m)} = \mathcal{M} \oplus \mathcal{K} \) where \( \mathcal{M} := \mathcal{K}^{(m-1)} \), and partition \( A \) as

\[
A = \begin{pmatrix}
B & C \\
0 & A_{m,m}
\end{pmatrix} \in \mathcal{L}(\mathcal{M} \oplus \mathcal{K}),
\]