Spectra for Factorable Matrices on $\ell^p$

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Abstract. We obtain the spectra and fine spectra for factorable matrices, considered as bounded linear operators over $\ell^p, 1 < p < \infty$.

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In this paper we study the spectrum and fine spectrum of matrices over $\ell^p$, $A = (a_{nk})$, of the type

$$a_{nk} = \begin{cases} a_n b_k, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Such matrices are called factorable. The spectrum and the fine spectrum of the Cesàro matrix, $p$-Cesàro matrices, Rhaly matrices and weighted mean matrices are determined by taking $a_n = 1/(n+1), b_k = 1$ ([7], [18], [11], [9], [10]), $a_n = \frac{1}{(n+1)^p}$ where $p > 1$, $b_k = 1$ ([13], [8]), ( $a_n = a_n, b_k = 1$ [12], [19], [20], [21], [22]) and $b_k = p_k, a_n = 1/P_n$ where $P_n = \sum_{k=0}^{n} p_k$ ([4], [3], [14], [15], [16], [17]), respectively.

Grahame Bennett in ([2], Theorem1) has obtained three necessary and sufficient conditions for a factorable matrix to be a bounded operator on $\ell^p$ for $1 < p < \infty$. They are:

(i) there exists $K_1$ such that, for $m = 1, 2, \ldots$,

$$\sum_{n=1}^{m} \left( a_n \sum_{k=1}^{n} b_k^p \right)^p \leq K_1 \left( \sum_{k=1}^{m} b_k^p \right);$$

(ii) there exists $K_2$ such that, for $m = 1, 2, \ldots$,

$$\left( \sum_{n=m}^{\infty} a_n^p \right)^{1/p} \left( \sum_{k=1}^{m} b_k^{p^*} \right)^{1/p^*} \leq K_2;$$
Thus there exists \( K_1 \) such that, for \( m = 1, 2, \ldots \),
\[
\sum_{k=m}^{\infty} \left( b_k \sum_{n=k}^{\infty} a_n^p \right)^p \leq K_1 \left( \sum_{n=m}^{\infty} a_n^p \right).
\]

He actually has necessary and sufficient conditions for a factorable matrix \( A : \ell^p \to \ell^q \). The above conditions are obtained by setting \( p = q \) in his theorem.

An infinite matrix \( A \) is said to be conservative if it is a selfmap of \( c \), the space of convergent sequences. Define \( f_\gamma \) by \( f_\gamma = \lim_{n \to \infty} a_{nk} \) for each \( k \), \( t_n := \sum_{k=0}^{\infty} a_{nk} \). A matrix \( A \) is said to be coregular if it is a selfmap of \( A \) and \( \chi(A) := \lim t_n - \sum f_k \neq 0 \) and if \( \chi(A) = 0 \), then \( A \) is called conull. Let \( c_n := a_n b_n \), \( t_n := a_n \sum_{k=0}^{\infty} b_k \), \( \lim n t_n = a \).

A matrix \( A \in B(\ell^\infty) \) if it maps every bounded sequence into a bounded sequence. Every conservative matrix \( A \in B(\ell^\infty) \).

Define \( \gamma := \lim c_n \).

**Theorem 1.** Let \( A \) be a coregular factorable lower triangular matrix with nonnegative entries such that \( \gamma > 0 \). Then \( A \in B(\ell^p) \) \((1 < p < \infty)\).

**Proof.** Since \( A \in B(\ell^\infty) \), according to the Riesz-Thorin Theorem, to show that \( A \in B(\ell^p) \) \((1 < p < \infty)\), it is enough to show that \( A \in B(\ell^1) \).

Since \( \gamma > 0 \), there exists at least one \( N_1 \) such that \( c_n \geq \frac{\gamma}{2} \) each \( n \geq N_1 \).

For \( n \geq N_1 \),
\[
\frac{a_{n+1}}{a_n} = \frac{t_{n+1}}{t_n} \frac{\sum_{k=0}^{n+1} b_k}{b_{n+1}} \leq \frac{t_{n+1}}{t_n} \left( 1 - \frac{b_{n+1}}{\sum_{k=0}^{n+1} b_k} \right) \leq \frac{t_{n+1}}{t_n} \left( 1 - \frac{\gamma}{2t_{n+1}} \right) \to 1 - \frac{\gamma}{2\alpha} \text{ as } n \to \infty.
\]

Thus there exists an \( N_2 \geq N_1 \) such that, for all \( n \geq N_2 \),
\[
\frac{a_{n+1}}{a_n} \leq a - \frac{\gamma}{4\alpha}.
\]

For all \( k \geq N_2 \),
\[
b_k a_{k+j} = b_k a_k \frac{a_{k+1}}{a_k} \ldots \frac{a_{k+j-1}}{a_{k+j-2}} \frac{a_{k+j}}{a_{k+j-1}} \leq a_k b_k \prod_{j=k}^{k+j-1} \frac{a_{j+1}}{a_j} \leq a_k b_k \prod_{j=k}^{k+j-1} \left( 1 - \frac{\gamma}{4\alpha} \right) = a_k b_k \left( 1 - \frac{\gamma}{4\alpha} \right)^j,
\]