On Bounded Local Resolvents

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Abstract. It is known that each normal operator on a Hilbert space with non-empty interior of the spectrum admits vectors with bounded local resolvent. We generalize this result for Banach space operators with the decomposition property (δ) (in particular for decomposable operators). Moreover, the same result holds for operators with interior points in the localizable spectrum.

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Let $X$ be a complex Banach space and $B(X)$ the Banach algebra of all bounded linear operators on $X$. Let $T \in B(X)$. It is well known that the resolvent mapping $(T - z)^{-1}$, which is defined and analytic on the resolvent set $\rho(T)$, is unbounded. On the other hand, the behavior of local resolvent functions may be quite different. In [1], Bermúdez and González have shown that a normal operator $N$ on a separable Hilbert space has a non-trivial bounded local resolvent function if and only if the interior of the spectrum of $N$ is not empty, i.e., $\text{Int} \sigma(N) \neq \emptyset$. Neumann [4] extended this result to non-separable spaces, and proved a similar result for multiplication operators induced by a given continuous function on the Banach algebra $\mathbb{C}(\Omega)$ of all continuous complex-valued functions on a compact Hausdorff space $\Omega$.

In this article we show that there is a quite large class of bounded operators on a complex Banach space that have non-trivial bounded local resolvent functions. In particular, every decomposable operator $T$ with $\text{Int} \sigma(T) \neq \emptyset$ has this property. On the other hand, there is a decomposable operator $T$ with $\text{Int} \sigma(T) = \emptyset$, which admits a local resolvent function that is not only bounded but can be even continuously extended to the whole complex plane.

Before we state our main results we are going to introduce some notation and terminology from local spectral theory (the reader is referred to [3] for details).
An operator \( T \in B(X) \) is said to have the single-valued extension property (SVEP) if, for every open set \( U \subseteq \mathbb{C} \), the only analytic solution \( f : U \to X \) of the equation
\[
(T - z)f(\lambda) = 0 \quad (z \in U)
\]
is the function \( f \equiv 0 \).

The local resolvent set \( \rho_T(x) \) of an operator \( T \) with SVEP at \( x \in X \) is defined as the set of all \( w \in \mathbb{C} \), for which there exists an analytic function \( f : U \to X \) on an open neighbourhood \( U \) of \( w \) such that \( (T - z)f(z) = x \) for all \( z \in U \). Let
\[
f(z) = \sum_{i=1}^{\infty} x_i (z - w)^i \quad (z \in U)
\]
be the Taylor expansion of \( f \). Comparing the coefficients, it is easy to see that \( w \in \rho_T(x) \) if and only if there are vectors \( x_1, x_2, \ldots \in X \) such that \( (T - w)x_{i+1} = x_i \ (i \geq 1) \). It is easy to see that \( \rho_T(x) \) is always closed; if \( \rho_T(x) \) is nonempty. It is easy to see that \( \sigma_T(x) = \mathbb{C} \setminus \rho_T(x) \) is called the local spectrum of \( T \) at \( x \).

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Proof. Choose a sequence \( (\lambda_n)_{n=1}^{\infty} \subseteq U \) which is dense in \( U \) and such that \( \lambda_i \neq \lambda_j \ (i \neq j) \). We shall construct a sequence of vectors \( (x_n)_{n=1}^{\infty} \subseteq X \) such that \( \lambda_n \in \sigma_T(x_n) \subseteq U \), \( x_n \notin \text{Im} (T - \lambda_1) \), and \( x_n \in \text{Im} (T - \lambda_j) \), for all \( 1 \leq j < n \).

Let \( n \in \mathbb{N} \). The property SVEP implies that \( \text{Im} (T - \lambda_1) \neq X \) ([3], Proposition 1.3.2 (f)). Choose \( u \in X \setminus \text{Im} (T - \lambda_1) \). Let \( V \) and \( V' \) be open sets such that
\[
\lambda_n \in V \subseteq U' \subseteq U' \subseteq U.
\]