\( H^p_w - L^p_w \) Boundedness of Marcinkiewicz Integral

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Abstract. The Marcinkiewicz integral is essentially a Littlewood-Paley \( g \)-function, which plays a very important role in harmonic analysis. In this paper we give weaker smoothness conditions assumed on \( \Omega \) to imply the \( H^p_w - L^p_w \) boundedness of the Marcinkiewicz integral operator \( \mu_\Omega \), where \( w \) belongs to the Muckenhoupt weight class.

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1. Introduction

Let \( S^{n-1} \) denote the unit sphere in \( \mathbb{R}^n \) \((n \geq 2)\) and \( d\sigma \) be the Lebesgue measure on \( S^{n-1} \). Let \( \Omega \) be a homogeneous function of degree zero on \( \mathbb{R}^n \) which is locally integrable and satisfies

\[
\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,
\]

where \( x' = x'/|x| \) for any \( x \neq 0 \). For a function \( f \) on \( \mathbb{R}^n \), the Marcinkiewicz integral operator \( \mu_\Omega \) is defined by

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty \left| F_{\Omega,t}(f,x) \right|^2 \frac{dt}{t^2} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(f,x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.
\]

The operator \( \mu_\Omega \) was originally introduced by Marcinkiewicz [11] in 1938 for \( n = 1 \) and \( \Omega(t) = \text{sign } t \). In 1958, Stein [12] defined the Marcinkiewicz integral of higher dimensions and proved that if \( \Omega \in \text{Lip}_\alpha(S^{n-1}) \), \( 0 < \alpha \leq 1 \), then \( \mu_\Omega \) is of type \( (p,p) \) for \( 1 < p \leq 2 \) and of weak type \( (1,1) \). In 1962, Benedek, Calderón and

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Panzone [1] showed that if $\Omega \in C^1(S^{n-1})$, then $\mu_\Omega$ is of type $(p, p)$ for $1 < p < \infty$. In 1990, Torchinsky and Wang [14] proved that if $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then, for $1 < p < \infty$ and $w \in A_p$ (the Muckenhoupt weight class), $\mu_\Omega$ is bounded on $L^p_w$. It is worth pointing out that the results mentioned above were obtained when $\Omega$ satisfies some smoothness conditions.

In 1999, Ding, Fan, and Pan [3] improved Torchinsky and Wang’s result by ridding of the smoothness condition assumed on $\Omega$.

**Theorem 1.1.** Let $\Omega \in L^q(S^{n-1})$, $1 < q < \infty$. If $w^\beta \in A_p$, $1 < p < \infty$, then there is a constant $C > 0$ independent of $f$ such that

$$\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{L^q_w}.$$  

As to the $H^p - L^p$ boundedness, recently Ding, Lee, and Lin [4] extended to the weighted case.

**Theorem 1.2.** Let $\Omega$ satisfy the $L^q$-Dini condition for $q > 1$. If $w^\beta \in A_1$, then there exists a constant $C > 0$ independent of $f$ such that $\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{H^q_w}$.

**Theorem 1.3.** Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$, and $\frac{\alpha}{\alpha+\beta} < p < 1$. If $\Omega \in \text{Lip}_\alpha$ and $w \in A_{p+\frac{\alpha}{\alpha+\beta}}$, then there exists a constant $C > 0$ independent of $f$ such that $\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{H^q_w}$.

In this article, we will show that under weaker smoothness conditions assumed on $\Omega$, which is called Din\(_{\alpha}(S^{n-1})\) and will be defined in the next section, the Marcinkiewicz integral operator $\mu_\Omega$ is bounded from $H^p_w$ to $L^p_w$.

**Theorem 1.4.** Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, \frac{1}{2}\}$, and $\frac{\alpha}{\alpha+\beta} < p < 1$. Suppose that $\Omega \in L^q(S^{n-1}) \cap \text{Din}_1^\alpha(S^{n-1})$ for $1 < q \leq \infty$. If

(a) $1 < q \leq \frac{1}{p}$ and $w^\beta \in A_{\frac{\alpha}{\alpha+\beta}}$; or

(b) $\frac{1}{p} < q < \infty$ and $w^{1/(1-p)} \in A_{\frac{\alpha}{\alpha+\beta}}$,

then there exists a constant $C > 0$ independent of $f$ such that $\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{H^q_w}$.

**Theorem 1.5.** Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$, and $\frac{\alpha}{\alpha+\beta} < p \leq 1$. Suppose that $\Omega \in \text{Din}_1^\alpha(S^{n-1})$ for $1 < q < \infty$. If $w^\beta \in A_{\frac{\alpha}{\alpha+\beta}}$, then there exists a constant $C > 0$ independent of $f$ such that $\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{H^q_w}$.

**Remark 1.6.** It is worthy noting that Theorem 1.2 can be regarded as the limit case of Theorem 1.5 by choosing $p = 1$ and letting $\alpha \to 0$.

We do have a substantial improvement of Theorems 1.3 and 1.2.

**Corollary 1.7.** Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$, and $\frac{\alpha}{\alpha+\beta} < p \leq 1$. Suppose that $\Omega \in \text{Din}_1^\alpha(S^{n-1})$. If $w \in A_{p+\frac{\alpha}{\alpha+\beta}}$, then there exists a constant $C > 0$ independent of $f$ such that $\|\mu_\Omega(f)\|_{L^q_w} \leq C\|f\|_{H^q_w}$.