Characteristic Functions for Ergodic Tuples
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Abstract. Motivated by a result on weak Markov dilations, we define a notion of characteristic function for ergodic and coisometric row contractions with a one-dimensional invariant subspace for the adjoints. This extends a definition given by G. Popescu. We prove that our characteristic function is a complete unitary invariant for such tuples and show how it can be computed.

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0. Introduction
If $Z = \sum_{i=1}^{d} A_i \cdot A_i^*$ is a normal, unital, ergodic, completely positive map on $B(H)$, the bounded linear operators on a complex separable Hilbert space, and if there is a (necessarily unique) invariant vector state for $Z$, then we also say that $\mathcal{A} = (A_1, \ldots, A_d)$ is a coisometric, ergodic row contraction with a one-dimensional invariant subspace for the adjoints. Precise definitions are given below. This is the main setting to be investigated in this paper.

In Section 1 we give a concise review of a result on the dilations of $Z$ obtained by R. Gohm in [7] in a chapter called ‘Cocycles and Coboundaries’. There exists a conjugacy between a homomorphic dilation of $Z$ and a tensor shift, and we emphasize an explicit infinite product formula that can be obtained for the intertwining unitary. [7] may also be consulted for connections of this topic to a scattering theory for noncommutative Markov chains by B. Kümmerer and H. Maassen (cf. [9]) and more general for the relevance of this setting in applications.

In this work we are concerned with its relevance in operator theory and correspondingly in Section 2 we shift our attention to the row contraction $\mathcal{A} = (A_1, \ldots, A_d)$. Our starting point has been the observation that the intertwining unitary mentioned above has many similarities with the notion of characteristic function occurring in the theory of functional models of contractions, as initiated
by B. Sz.-Nagy and C. Foias (cf. [11, 6]). In fact, the center of our work is the commuting diagram 3.3 in Section 3, which connects the results in [7] mentioned above with the theory of minimal isometric dilations of row contractions by G. Popescu (cf. [12]) and shows that the intertwining unitary determines a multi-analytic inner function, in the sense introduced by G. Popescu in [14, 15]. We call this inner function the extended characteristic function of the tuple $\mathcal{A}$, see Definition 3.3.

Section 4 is concerned with an explicit computation of this inner function. In Section 5 we show that it is an extension of the characteristic function of the $*$-stable part $\mathring{\mathcal{A}}$ of $\mathcal{A}$, the latter in the sense of Popescu's generalization of the Sz.-Nagy-Foias theory to row contractions (cf. [13]). This explains why we call our inner function an extended characteristic function. The row contraction $\mathring{\mathcal{A}}$ is a one-dimensional extension of the $*$-stable row contraction $\mathring{\mathcal{A}}$, and in our analysis we separate the new part of the characteristic function from the part already given by Popescu.

G. Popescu has shown in [13] that for completely non-coisometric tuples, in particular for $*$-stable ones, his characteristic function is a complete invariant for unitary equivalence. In Section 6 we prove that our extended characteristic function does the same for the tuples $\mathcal{A}$ described above. In this sense it is characteristic. This is remarkable because the strength of Popescu’s definition lies in the completely non-coisometric situation while we always deal with a coisometric tuple $\mathcal{A}$. The extended characteristic function also does not depend on the choice of the decomposition $\sum_{i=1}^{d} A_i \cdot A_i^*$ of the completely positive map $Z$ and hence also characterizes $Z$ up to conjugacy. We think that together with its nice properties established earlier this clearly indicates that the extended characteristic function is a valuable tool for classifying and investigating such tuples respectively such completely positive maps.

Section 7 contains a worked example for the constructions in this paper.

1. Weak Markov dilations and conjugacy

In this section we give a brief and condensed review of results in [7], Chapter 2, which will be used in the following and which, as described in the introduction, motivated the investigations documented in this paper. We also introduce notation.

A theory of weak Markov dilations has been developed in [2]. For a (single) normal unital completely positive map $Z : B(\mathcal{H}) \to B(\mathcal{H})$, where $B(\mathcal{H})$ consists of the bounded linear operators on a (complex, separable) Hilbert space, it asks for a normal unital $^*$-endomorphism $\tilde{J} : B(\mathcal{H}) \to B(\tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is a Hilbert space containing $\mathcal{H}$, such that for all $n \in \mathbb{N}$ and all $x \in B(\mathcal{H})$

$$Z^n(x) = p_{\tilde{\mathcal{H}}} \tilde{J}^n(xp_{\mathcal{H}}) |_{\mathcal{H}}.$$  

Here $p_{\mathcal{H}}$ is the orthogonal projection onto $\mathcal{H}$. There are many ways to construct $\tilde{J}$. In [7], 2.3, we gave a construction analogous to the idea of ‘coupling to a shift’