Lax-Phillips Scattering for Automorphic Functions Based on the Eisenstein Transform

Yoichi Uetake

Abstract. We construct a Lax-Phillips scattering system on the arithmetic quotient space of the Poincaré upper half-plane by the full modular group, based on the Eisenstein transform. We identify incoming and outgoing subspaces in the ambient space of all functions with finite energy-form for the non-Euclidean wave equation. The use of the Eisenstein transform along with some properties of the Eisenstein series of two variables enables one to work only on the space corresponding to the continuous spectrum of the Laplace-Beltrami operator. It is shown that the scattering matrix is the complex function appearing in the the functional equation of the Eisenstein series of two variables. We obtain a compression operator constructed from the Laplace-Beltrami operator, whose spectrum consists of eigenvalues that coincide, counted with multiplicities, with the non-trivial zeros of the Riemann zeta-function. For this purpose we construct and use a scattering model on the one-dimensional Euclidean space.

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1. Introduction

In this paper we construct a Lax-Phillips scattering system for automorphic functions on the fundamental domain $\mathcal{F}$ of the full modular group $SL_2(\mathbb{Z})$, based on the Eisenstein transform and some properties of the Eisenstein series of two variables. The Eisenstein transform is considered to be an analog of the Fourier transform for the continuous spectrum of the Laplace-Beltrami operator on the fundamental domain.

Since a scattering theoretic view of the theory of automorphic functions was suggested by Gelfand [3] in 1962, Pavlov and Faddeev [17] carried out this in 1972
by revealing an intimate connection between the harmonic analysis of automorphic functions and the Lax-Phillips scattering theory for the non-Euclidean wave equation. In the preface to his selected works [18, p. 210] Piatetski-Shapiro recollects: ‘In the course of our discussions, Gelfand realized the connection between scattering theory and asymptotic properties of Eisenstein series (later on, [B. S. Pavlov and] L. D. Faddeev wrote a paper on this).’ This was taken up and further studied by Lax and Phillips and culminated in their monograph [7] and its important supplement [8]. See also a recent exposition by Lax [6, §9.7, p. 154–163].

Roughly speaking, the scattering matrix $S(s)$, which compares the asymptotic behavior of incoming and outgoing waves, has the form of inner product $\langle (sI - A)^{-1}b, c \rangle + d$ in our case. Describing a given $S(s)$ in this form, called a weak resolvent in Nordgren, Radjavi and Rosenthal [15], has been well studied in system theory as a minimal realization problem. It can be shown that the poles of the scattering matrix coincide with the eigenvalues of $A$. See e.g., Uetake [20].

This type of function appears in the Nagy-Foias operator model theory [13] as a characteristic function, whose scattering theoretic interpretation was given by Adamjan and Arov. Connections with system theory were discovered and studied by Helton. For all these connections see e.g., Helton [4].

In §2 we redo a part of these theories for a simple scattering system on the one-dimensional Euclidean space. We call this a scattering model. Our model treats a slightly more general situation than the one mentioned above, namely the case where incoming and outgoing subspaces are not necessarily orthogonal to each other. We only treat the case where the scattering matrix is a scalar meromorphic complex function. Thus one can prove the pole correspondence between the scattering matrix and the resolvent of the generator of the Lax-Phillips semiflow, by using a complex function theory in Hardy space. We can avoid the explicit use of the notion of weak resolvent, since our main concern here is the correspondence of the spectrum of $A$ with the poles of $S(s)$, not being existence of the above $b, c, d$. We will use this scattering model to prove the pole correspondence in the automorphic scattering system in §3.

Our construction of scattering for automorphic functions is done in §3. In our construction of a Lax-Phillips scattering system in the fundamental domain of $SL_2(\mathbb{Z})$, we combine both analytical and algebraic tools. Analytic tools are $L^2$-spaces and linear estimates in these spaces. By algebraic ones we mean the use of scattering models, properties of the Eisenstein series, the Eisenstein transform, and explicit solutions. We hope that our construction of scattering for automorphic functions contains some new technical and conceptual points in that our construction is done directly on the continuous spectrum part in a way more faithful to the properties of the Eisenstein series. Namely we use the following properties of the Eisenstein series of two variables: (i) it satisfies the functional equation; (ii) it is a (non $L^2$-eigenfunction of the non-Euclidean Laplacian; (iii) it defines the so-called Eisenstein transform (see §3.2). In particular the use of (iii) is most characteristic of our construction. The Eisenstein series is an analog of the integral kernel $e^{-i\xi \tau}$ (of two variables $\xi$ and $\tau$) of the Fourier transform, which itself is