Scattering Operators for Matrix Zakharov-Shabat Systems

Francesco Demontis and Cornelis van der Mee

Abstract. In this article the scattering matrix pertaining to the defocusing matrix Zakharov-Shabat system on the line is related to the scattering operator arising from time-dependent scattering theory. Further, the scattering data allowing for a unique retrieval of the potential in the defocusing matrix Zakharov-Shabat system are characterized.

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1. Introduction

Consider the matrix Zakharov-Shabat system

\[-iJ \frac{dX(x, \lambda)}{dx} - V(x)X(x, \lambda) = \lambda X(x, \lambda), \quad x \in \mathbb{R},\]

where

\[J = \begin{pmatrix} I_n & 0_{n \times m} \\ 0_{m \times n} & -I_m \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0_{n \times n} & iq(x) \\ \mp iq(x)^\dagger & 0_{m \times m} \end{pmatrix},\]

\[I_p\] is the identity matrix of order \(p\), the dagger stands for the conjugate transpose, and the entries of \(q(x)\) belong to \(L^1(\mathbb{R})\). The plus sign in (2) occurs in the focusing case and the minus sign in the defocusing case. Equation (1) has been studied extensively. We mention the original articles by Zakharov and Shabat [25] (\(n = m = 1\)) and Manakov [15] (\(n = 1\) and \(m = 2\)) and in particular [1, 2, 3], where also some of the applications are discussed. For the applications to fiber optics we...
refer to [13, 21]. Equation (1) can also be viewed as a so-called canonical system (cf. [20, 5] and references therein).

As in [4, 22, 8], for \( \lambda \in \mathbb{R} \) we define the Jost solution from the left, \( F_l(x, \lambda) \), and the Jost solution from the right, \( F_r(x, \lambda) \), as the \((n+m) \times (n+m)\) matrix solutions of (1) that satisfy the boundary conditions

\[
F_l(x, \lambda) = e^{i\lambda Jx} [I + o(1)], \quad x \to +\infty, \\
F_r(x, \lambda) = e^{i\lambda Jx} [I + o(1)], \quad x \to -\infty.
\]

Actually, the Jost solutions follow from the Volterra integral equations

\[
F_l(x, \lambda) = e^{i\lambda Jx} - iJ \int_x^{\infty} dy e^{i\lambda J(y-x)} V(y) F_l(y, \lambda), \\
F_r(x, \lambda) = e^{i\lambda Jx} + iJ \int_{-\infty}^{x} dy e^{i\lambda J(y-x)} V(y) F_r(y, \lambda).
\]

Then these two Jost solutions satisfy

\[
F_l(x, \lambda) = e^{i\lambda Jx} [a_l(\lambda) + o(1)], \quad x \to -\infty, \\
F_r(x, \lambda) = e^{i\lambda Jx} [a_r(\lambda) + o(1)], \quad x \to +\infty,
\]

where \( a_l(\lambda) \) and \( a_r(\lambda) \) are called transition matrices. It is easily seen that \( a_l(\lambda) \) and \( a_r(\lambda) \) are each other’s inverses, while

\[
a_l(\lambda)^{-1} = J a_l(\lambda)^\dagger J, \quad a_r(\lambda)^{-1} = J a_r(\lambda)^\dagger J, \quad \text{defocusing case}, \\
a_l(\lambda)^{-1} = a_l(\lambda)^\dagger, \quad a_r(\lambda)^{-1} = a_r(\lambda)^\dagger, \quad \text{focusing case}.
\]

Moreover, since (1) is a first order system, we have

\[
F_l(x, \lambda) = F_r(x, \lambda) a_l(\lambda), \quad F_r(x, \lambda) = F_l(x, \lambda) a_r(\lambda).
\]

Introducing the Faddeev functions

\[
M_l(x, \lambda) = F_l(x, \lambda) e^{-i\lambda Jx}, \quad M_r(x, \lambda) = F_r(x, \lambda) e^{-i\lambda Jx},
\]

we obtain from (4) the Volterra integral equations

\[
M_l(x, \lambda) = I_{n+m} - iJ \int_x^{\infty} dy e^{i\lambda J(y-x)} V(y) M_l(y, \lambda) e^{i\lambda J(y-x)}, \\
M_r(x, \lambda) = I_{n+m} + iJ \int_{-\infty}^{x} dy e^{i\lambda J(y-x)} V(y) M_r(y, \lambda) e^{i\lambda J(y-x)}.
\]

Defining the modified Faddeev functions

\[
m_+(x, \lambda) = M_l(x, \lambda) \left( I_n + 0_{m \times m} \right) + M_r(x, \lambda) \left( 0_{n \times n} + I_m \right), \\
m_-(x, \lambda) = M_l(x, \lambda) \left( I_n + 0_{m \times m} \right) + M_r(x, \lambda) \left( 0_{n \times n} + I_m \right),
\]

where \( A + B \) denotes the direct sum of the square matrices \( A \) and \( B \), we see that \( m_+(x, \lambda) \) is analytic in \( \lambda \in \mathbb{C}^+ \) and continuous in \( \lambda \in \mathbb{C}^+ \) and \( m_-(x, \lambda) \) is analytic

\footnote{In [3] the term “Jost function” is used for the \((n+m) \times n\) and \((n+m) \times m\) submatrices composed of the first \( n \) and last \( m \) columns, respectively.}