On the Right (Left) Invertible Completions for Operator Matrices

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Abstract. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces, and let $A \in B(\mathcal{H}_1)$, $B \in B(\mathcal{H}_2)$ and $C \in B(\mathcal{H}_2, \mathcal{H}_1)$ be given operators. A necessary and sufficient condition is given for $(\frac{AC}{X})$ to be a right (left) invertible operator for some $X \in B(\mathcal{H}_1, \mathcal{H}_2)$. Furthermore, some related results are obtained.

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1. Introduction

The study of operator matrices arises naturally from the following fact: if $\mathcal{H}$ is a Hilbert space and we decompose $\mathcal{H}$ as a direct sum of two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, each bounded linear operator $T: \mathcal{H} \to \mathcal{H}$ can be expressed as the operator matrix form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with respect to the space decomposition, where $T_{ij}$ is an operator from $\mathcal{H}_j$ into $\mathcal{H}_i$, $i,j = 1,2$. One way to study operators is to see them as being composed of simpler operators. The operator matrices have been studied by numerous authors [4–6,8,9,12,14–19,23,25]. This paper is concerned with the right (left) invertibility of $2 \times 2$ operator matrices.

In this paper, $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces. Let $B(\mathcal{H}_1, \mathcal{H}_2)$ and $K(\mathcal{H}_1, \mathcal{H}_2)$ denote the sets of bounded linear operators and compact operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, respectively. When $\mathcal{H}_1 = \mathcal{H}_2$ we write $B(\mathcal{H}_1, \mathcal{H}_1) = $.

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$B(\mathcal{H}_1)$. If $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, we use $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $T^*$ to denote the range space, the null space and the adjoint of $T$. For a linear subspace $\mathcal{M} \subset \mathcal{H}_1$, its closure and orthogonal complement are denoted by $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$. Write $P_{\overline{\mathcal{M}}}$ for the orthogonal projection onto $\overline{\mathcal{M}}$ along $\mathcal{M}^{\perp}$ and $T|_{\mathcal{M}}$ for the restriction of $T$ to $\mathcal{M}$.

Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$. Recall that a linear operator $T^+$ from $\mathcal{H}_2$ into $\mathcal{H}_1$ is said to be the Moore–Penrose generalized inverse of $T$ if $T^+$ satisfies $\mathcal{D}(T^+) = \mathcal{R}(T) \oplus \mathcal{R}(T)^{\perp}$ ($\mathcal{D}(T^+)$ denotes the domain of $T^+$) and the four Moore–Penrose equations:

$$TT^+T = T, \quad T^+T = I - P_{\mathcal{N}(T)},$$

$$T^+TT^+ = T^+, \quad TT^+ = P_{\overline{\mathcal{R}(T)}}|_{\mathcal{D}(T^+)}. $$

The Moore–Penrose generalized inverse $T^+$ is uniquely determined and is a closed linear operator. In particular, for any $y \in \mathcal{R}(T)$ we have $y = TT^+y$. From the closed graph theorem (see [24]), we know that $T^+$ is bounded if and only if $\mathcal{R}(T)$ is closed, and in this case, $\mathcal{D}(T^+) = \mathcal{H}_2$ (see [3,21]). The following properties of $T^+$ are well known (see [3,22]): If $\mathcal{R}(T)$ is closed, then

$$\mathcal{R}(T^+) = \mathcal{R}(T^*) = \mathcal{N}(T)^{\perp}, \quad \mathcal{N}(T^+) = \mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}. $$

An operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is called a right (respectively, left) invertible operator if there exists an operator $S \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that $TS = I_{\mathcal{H}_2}$ (respectively, $ST = I_{\mathcal{H}_1}$). If $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is both left invertible and right invertible, we call it invertible. It is well known [7] that $T$ is right invertible if and only if $T$ is surjective, i.e., $\mathcal{R}(T) = \mathcal{H}_2$. Also, $T$ is left invertible if and only if $\|Tx\| \geq c\|x\|$ for all $x \in \mathcal{H}_1$ and some constant $c > 0$, i.e., $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) = \{0\}$. The right (or defect) spectrum $\sigma_r(T)$ of $T \in B(\mathcal{H}_1)$ is defined by

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right invertible}\},$$

whilst the left (or approximate point) spectrum $\sigma_l(T)$ of $T \in B(\mathcal{H}_1)$ is defined by

$$\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left invertible}\}. $$

It is evident [1,2,7] that $\sigma_l(T)$ (respectively, $\sigma_r(T)$) is a compact nonempty subset of $\mathbb{C}$, and $\partial(\sigma_r(T) \cup \sigma_l(T)) \subseteq \sigma_l(T) \cap \sigma_r(T)$, where we write $\partial K$ for the topological boundary of a subset $K \subset \mathbb{C}$. We also have from [7] that $\lambda \in \sigma_l(T)$ if and only if $\lambda \in \sigma_r(T)^\ast$.

Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, $n(T) = \dim \mathcal{N}(T)$ and $d(T) = \dim \mathcal{R}(T)^{\perp}$. If $\mathcal{R}(T)$ is closed and $n(T) < \infty$, we call $T$ a left Fredholm operator (or upper semi-Fredholm operator), and if $\mathcal{R}(T)$ is closed and $d(T) < \infty$, then $T$ is called a right Fredholm operator (or lower semi-Fredholm operator) (see [1,17,18,20]).

Given an arbitrary operator $T \in B(\mathcal{H}_1)$, the right essential spectrum $\sigma_{re}(T)$ is defined by

$$\sigma_{re}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not right Fredholm}\},$$

and the left essential spectrum $\sigma_{le}(T)$ is defined by

$$\sigma_{le}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Fredholm}\}. $$