Topology and Smooth Structure for Pseudoframes

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Abstract. Given a closed subspace $S$ of a Hilbert space $H$, we study the sets $\mathcal{F}_S$ of pseudo-frames, $\mathcal{C}\mathcal{F}_S$ of commutative pseudo-frames and $\mathcal{X}_S$ of dual frames for $S$, via the (well known) one to one correspondence which assigns a pair of operators $(F, H)$ to a frame pair $(\{f_n\}_{n \in \mathbb{N}}, \{h_n\}_{n \in \mathbb{N}})$,

$$F : \ell^2 \to H, \quad F(\{c_n\}_{n \in \mathbb{N}}) = \sum_n c_n f_n,$$

and

$$H : \ell^2 \to H, \quad H(\{c_n\}_{n \in \mathbb{N}}) = \sum_n c_n h_n.$$

We prove that, with this identification, the sets $\mathcal{F}_S$, $\mathcal{C}\mathcal{F}_S$ and $\mathcal{X}_S$ are complemented submanifolds of $\mathcal{B}(\ell^2, H) \times \mathcal{B}(\ell^2, H)$. We examine in more detail $\mathcal{X}_S$, which carries a locally transitive action from the general linear group $GL(\ell^2)$. For instance, we characterize the homotopy theory of $\mathcal{X}_S$ and we prove that $\mathcal{X}_S$ is a strong deformation retract both of $\mathcal{F}_S$ and $\mathcal{C}\mathcal{F}_S$; therefore these sets share many of their topological properties.

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1. Introduction

Let $H$ be a Hilbert space, $\mathcal{B}(\ell^2, H)$ the Banach space of bounded (linear) operators acting from $\ell^2$ to $H$, and $S \subset H$ a closed subspace of $H$. In this paper we study the set

$$\mathcal{F}_S = \{(F, H) \in \mathcal{B}(\ell^2, H) \times \mathcal{B}(\ell^2, H) : FH^*|_S = id_S\}. \quad (1)$$

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If $Q$ is an idempotent with $R(Q) = S$, then
\[ \mathcal{F}_S = \{(F, H) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : FH^*Q = Q\}. \]

This set parametrizes the set of what S. Li and H. Ogawa call pseudoframes for the subspace $S$ [21]. Let us recall their definition: A pair of sequences $\{f_n\}_{n \in \mathbb{N}}, \{h_n\}_{n \in \mathbb{N}}$ of vectors in $\mathcal{H}$ is a pseudoframe for the subspace $S$ if

1. $\{f_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ are Bessel sequences of $\mathcal{H}$ (see definition in Sect. 2).
2. For any $f \in S$,
\[ f = \sum_{n=1}^{\infty} \langle f, h_n \rangle f_n. \]

As these authors remark, pseudoframes for $S$ are in one to one correspondence with pairs of bounded operators $F, H$, defined by
\[ H^* : \mathcal{H} \to \ell^2, \quad H^*x = (\langle x, h_n \rangle)_n, \]
and
\[ F : \ell^2 \to \mathcal{H}, \quad F\left(\{c_n\}_{n \in \mathbb{N}}\right) = \sum_n c_n f_n \]
such that $FH^*|_S = id_S$.

In fact, Li and Ogawa require less, namely that $\{f_n\}_{n \in \mathbb{N}}$ be a Bessel sequence with respect to $S$, which means that the condition $\sum_n |\langle f, f_n \rangle|^2 \leq c\|f\|^2$ holds only for $f \in S$. This weaker condition gives naturally representations using unbounded operators. We shall consider this setting elsewhere. Therefore we may call the present version bounded pseudoframes for $S$.

We shall use this one-to-one correspondence between bounded pseudoframes for the subspace $S$ and the set $\mathcal{F}_S \subset \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$ to endow the former with a topological structure, namely the one given by the operator norm in the space $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. This point of view is not new. There are many papers dealing with frames by examining their analysis and synthesis operators. Among them, let us mention [10, 12, 15, 20, 21]. What is gained by this abstract presentation of frames, is that different special sets of frames, as the ones considered below, are regarded as spaces in a common ambient Banach space. Questions about the local structure of these spaces can be examined, for instance, if any given pair of frames in one of these spaces can be joined by continuous or a smooth path inside the space.

As it turns out, $\mathcal{F}_S$ is a smooth complemented submanifold of the Banach space $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. We study its properties, as well as those of the following subsets:

1. The set of $C\mathcal{F}_S \subset \mathcal{F}_S$ of commutative pseudoframes [21], which consists of pairs $(F, H)$ in $\mathcal{F}_S$ which also verify that $PFH^* = P$ (where $P$ is the orthogonal projection onto $S$). This set parametrizes the set of pairs of sequences $(\{f_n\}_{n \in \mathbb{N}}, \{h_n\}_{n \in \mathbb{N}})$ such that
\[ f = \sum_{n=1}^{\infty} \langle f, h_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle h_n, \]