Some Closed Range Integral Operators on Spaces of Analytic Functions

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To Mary Rose

Abstract. Our main result is a characterization of $g$ for which the operator $S_g(f)(z) = \int_0^z f'(w)g(w)\,dw$ is bounded below on the Bloch space. We point out analogous results for the Hardy space $H^2$ and the Bergman spaces $A^p$ for $1 \leq p < \infty$. We also show the companion operator $T_g(f)(z) = \int_0^z f(w)g'(w)\,dw$ is never bounded below on $H^2$, Bloch, nor BMOA, but may be bounded below on $A^p$.

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1. Introduction

We examine operators on Banach spaces of analytic functions on the unit disk in the complex plane. The operator $T_g$, with symbol $g(z)$ an analytic function on the disk, is defined by

$$T_g f(z) = \int_0^z f(w)g'(w)\,dw.$$ 

$T_g$ is a generalization of the standard integral operator, which is $T_g$ when $g(z) = z$. Letting $g(z) = \log(1/(1-z))$ gives the Cesáro operator. Discussion of the operator $T_g$ first arose in connection with semigroups of composition operators. (see [11] for background) Characterizing the boundedness and compactness of $T_g$ on certain spaces of analytic functions is of recent

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interest, as seen in [1,2,5,11], and open problems remain. $T_g$ and its companion operator $S_g f(z) = \int_0^z f'(w)g(w) \, dw$ are related to the multiplication operator $M_g f(z) = g(z)f(z)$, since integration by parts gives

$$M_g f = f(0)g(0) + T_g f + S_g f.$$ 

If any two of $M_g$, $S_g$, and $T_g$ are bounded, then so is the third. But in some situations one operator is bounded while two are unbounded. Boundedness of $T_g$ on the Hardy and Bergman spaces and $BMOA$ is characterized in [1,2,11]. The pointwise multipliers of these and many other spaces are well known. See [12] for $BMOA$.

In this paper we examine the property of being bounded below for $T_g$ and $S_g$ on spaces of analytic functions. We examine aspects of the problems on Hardy and Bergman spaces, the Bloch space, and $BMOA$. In doing so we must assume the operators are bounded, and we study characterizations of the symbols for which the operators are bounded. Consideration of $M_g$ is useful as well.

2. Preliminaries

The notation $f \lesssim g$ will mean there exists a universal constant $C$ such that $f \leq C g$. $f \approx g$ will mean $f \lesssim g \lesssim f$.

Let $D$ be the unit disk in the complex plane. Let $H(D)$ denote the set of analytic functions on $D$. For $1 \leq p < \infty$, the Hardy space $H^p$ on $D$ is

$$H^p = \left\{ f \in H(D) : \| f \|_p = \sup_{0 < r < 1} \left( \frac{2\pi}{0} \int |f(re^{i\theta})|^p \, d\theta \right) < \infty \right\}.$$ 

The space of bounded analytic functions on $D$ is

$$H^\infty = \left\{ f \in H(D) : \| f \|_\infty = \sup_{z \in D} |f(z)| < \infty \right\}.$$ 

We define weighted Bergman spaces, for $\alpha > -1$,

$$A^p_\alpha = \left\{ f \in H(D) : \| f \|_{A^p_\alpha} = \int_D |f(z)|^p (1 - |z|^2)\alpha \, dA(z) < \infty \right\},$$

where $dA(z)$ refers to Lebesgue area measure on $D$.

The Bloch space is

$$B = \left\{ f \in H(D) : \| f \|_B = \sup_{z \in D} |f'(z)|(1 - |z|^2) < \infty \right\}.$$ 

Note that $\| \cdot \|_B$ is a semi-norm. The true norm accounts for functions differing by an additive constant.

A complex measure $\mu$ on $D$ is called a (Hardy space) Carleson measure if there exists $C > 0$ such that $\mu(S(I)) \leq C |I|$ for all arcs $I \subseteq \partial D$, where $S(I) = \{ re^{i\theta} : 1 - |I| < r < 1, e^{i\theta} \in I \}$ is the Carleson rectangle associated with $I$, and $|I|$ is the length of $I$. The smallest such $C$ is called the Carleson constant for the measure $\mu$. Define, for $f \in H(D)$, $d\mu_f(z) =$