Complex Symmetry of a Dense Class of Operators

Sen Zhu and Chun Guang Li

Abstract. In this paper, we develop new techniques to study complex symmetric operators. We first give an interpolation theorem related to conjugations. This result is used to give a geometric characterization for a norm-dense class of operators to be complex symmetric. Also we characterize certain complex symmetric nilpotent operators, and several illustrating examples are given.

Mathematics Subject Classification (2000). Primary 47A05; Secondary 47B37, 47B99.

Keywords. Complex symmetric operators, weighted shifts, nilpotent operators, compact operators.

1. Introduction

The main aim of this paper is to give a geometric characterization of a norm-dense class of operators being complex symmetric. This work is partially inspired by a paper of Garcia et al. [7] in which they characterize when a square complex matrix having distinct singular eigenvalues is unitarily equivalent to a complex symmetric matrix. Our main result is an infinite dimensional extension of their result. To proceed, let us first recall several notations and definitions.

Throughout this paper, we let $\mathbb{C}, \mathbb{N}$ and $\mathbb{Z}$ denote the set of complex numbers, the set of positive integers and the set of integers respectively. We always denote by $\mathcal{H}$ a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We let $\mathcal{K}(\mathcal{H})$ denote the ideal of all compact operators in $\mathcal{B}(\mathcal{H})$.

This work was supported by NNSF of China (11101177, 11026038, 10971079), China Postdoctoral Science Foundation (2011M500064) and the Basic Research Foundation of Jilin University (201001001, 201103194).
Definition 1.1. A conjugation on $H$ is a conjugate-linear map $C : H \to H$ satisfying that $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in H$.

Definition 1.2. We say that an operator $T \in \mathcal{B}(H)$ is complex symmetric, denoted by $T \in (cs)$, if there exists a conjugation $C$ on $H$ so that $CTC = T^*$. In this case, $T$ is said to be $C$-symmetric. We denote by $\mathcal{S}(H)$ the set of all complex symmetric operators on $H$.

Garcia and Putinar [8] initiated the study of complex symmetric operators (CSOs, for short), which have many motivations in function theory, matrix analysis and other areas. Some important results concerning the internal structure of CSOs have been obtained (see [1–3,5–10,12–14,17,18,20] for references).

An effective way to investigate the structure of CSOs is to characterize which special operators are complex symmetric. In fact, a lot of previous work concerning CSOs focuses on this basic question. Many important classes of operators such as compact operators, weighted shifts and partial isometries are studied [8,12,17,20]. All these results indicate that even in finite dimensional Hilbert spaces it is often difficult to verify if a given operator is complex symmetric (see [1,11]). It is natural to ask the following question: Is it possible to give a clear and complete characterization of when a general operator is complex symmetric? In view of the previous results, maybe the answer is “not”. So we are devoted to solving the following problem.

Problem 1.3. Give a clear and concise characterization of the complex symmetry for as many operators as possible.

The main aim of this paper is to give a geometric characterization of the following operators $T \in \mathcal{B}(H)$ to be complex symmetric (Theorem 2.3):

$$ T = \sum_{i \in \Lambda} a_i f_i \otimes e_i, \quad (1.1) $$

where $\Lambda \subset \mathbb{N}, a_i > 0$ for $i \in \Lambda, \{e_i\}_{i \in \Lambda}$ and $\{f_i\}_{i \in \Lambda}$ are two orthonormal subsets of $H$. In particular, when $\{a_i\}_{i \in \Lambda}$ are pairwise distinct, the characterization of $T$ being complex symmetric is more explicit. The main result of this paper is the following theorem.

Theorem 1.4. (Main Theorem) Assume that $\{e_i\}_{i \in \Lambda}, \{f_i\}_{i \in \Lambda}$ are two orthonormal subsets of $H$ and $T \in \mathcal{B}(H)$ can be written as

$$ T = \sum_{i \in \Lambda} a_i f_i \otimes e_i, \quad (1.2) $$

where $a_i > 0$ and $a_i \neq a_j$ for all $i, j \in \Lambda$ with $i \neq j$. Then the following are equivalent.

(i) $T$ is complex symmetric.

(ii) There exists a sequence $\{\lambda_i\}_{i \in \Lambda}$ of complex numbers with $|\lambda_i| = 1$ for all $i \in \Lambda$ such that $\lambda_i \langle e_i, f_j \rangle = \lambda_j \langle e_j, f_i \rangle$ for all $i, j \in \Lambda$.

(iii) $|\langle e_m, f_n \rangle| = |\langle e_n, f_m \rangle|$ for all $m, n \in \Lambda$ and

$$ \langle e_i, f_j \rangle \langle e_j, f_k \rangle \langle e_k, f_i \rangle = \langle e_i, f_k \rangle \langle e_k, f_j \rangle \langle e_j, f_i \rangle $$

for any triad $(i, j, k)$ in $\Lambda$ with $i \leq j \leq k$. 