New Hardy Spaces of Musielak–Orlicz Type and Boundedness of Sublinear Operators

Luong Dang Ky

Abstract. We introduce a new class of Hardy spaces $H^{\varphi(\cdot, \cdot)}(\mathbb{R}^n)$, called Hardy spaces of Musielak–Orlicz type, which generalize the Hardy–Orlicz spaces of Janson and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky. Here, $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a function such that $\varphi(x, \cdot)$ is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt $A_{\infty}$ weight. A function $f$ belongs to $H^{\varphi(\cdot, \cdot)}(\mathbb{R}^n)$ if and only if its maximal function $f^*$ is so that $x \mapsto \varphi(x, |f^*(x)|)$ is integrable. Such a space arises naturally for instance in the description of the product of functions in $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ respectively (see Bonami et al. in J Math Pure Appl 97:230–241, 2012). We characterize these spaces via the grand maximal function and establish their atomic decomposition. We characterize also their dual spaces. The class of pointwise multipliers for $BMO(\mathbb{R}^n)$ characterized by Nakai and Yabuta can be seen as the dual of $L^1(\mathbb{R}^n) + H^{\log(\cdot)}(\mathbb{R}^n)$ where $H^{\log(\cdot)}(\mathbb{R}^n)$ is the Hardy space of Musielak–Orlicz type related to the Musielak–Orlicz function $\theta(x,t) = \frac{t}{\log(e+|x|+t) + \log(e+t)}$. Furthermore, under additional assumption on $\varphi(\cdot, \cdot)$ we prove that if $T$ is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $B$, then $T$ uniquely extends to a bounded sublinear operator from $H^{\varphi(\cdot, \cdot)}(\mathbb{R}^n)$ to $B$. These results are new even for the classical Hardy–Orlicz spaces on $\mathbb{R}^n$.

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1. Introduction

Since Lebesgue theory of integration has taken a center stage in concrete problems of analysis, the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$-families naturally arose. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$ are good substitutes of $L^p(\mathbb{R}^n)$ when studying the boundedness of operators: for example, the Riesz operators are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. The theory of Hardy spaces
$H^p$ on the Euclidean space $\mathbb{R}^n$ was initially developed by Stein and Weiss [52]. Later, Fefferman and Stein [19] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which now plays an important role in various fields of analysis and partial differential equations; see, for example, [15, 16, 44]. A key feature of the classical Hardy spaces is their atomic decomposition characterizations, which were obtained by Coifman [14] when $n = 1$ and Latter [36] when $n > 1$. Later, the theory of Hardy spaces and their dual spaces associated with Muckenhoupt weights have been extensively studied by García-Cuerva [21], Strömberg and Torchinsky [53] (see also [10, 22, 43]); there the weighted Hardy spaces was defined by using the nontangential maximal functions and the atomic decompositions were derived. On the other hand, as another generalization of $L^p(\mathbb{R}^n)$, the Orlicz spaces were introduced by Birnbaum–Orlicz in [3] and Orlicz in [48], since then, the theory of the Orlicz spaces themselves has been well developed and the spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [2, 28, 39]. Moreover, the Hardy–Orlicz spaces are also good substitutes of the Orlicz spaces in dealing with many problems of analysis, say, the boundedness of operators.

Let $\Phi$ be a Orlicz function which is of positive lower type and (quasi-)concave. In [32], Janson has considered the Hardy–Orlicz space $H^\Phi(\mathbb{R}^n)$ the space of all tempered distributions $f$ such that the nontangential grand maximal function $f_t^*(x) = \sup_{y \in B(x,t)} |f_t(y)|$, for all $x \in \mathbb{R}^n$, here and in what follows $f_t(x) := t^{-n}f(t^{-1}x)$, with

$$A_N = \{ \phi \in S(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial_\alpha^\phi(x)| \leq 1 \text{ for } \alpha \in \mathbb{N}^n, |\alpha| \leq N \}$$

with $N = N(n, \Phi)$ taken large enough, belongs to the Orlicz space $L^\Phi(\mathbb{R}^n)$. Recently, the theory of Hardy–Orlicz spaces associated with operators (see [12, 13, 33, 58]) have also been introduced and studied. Remark that these Hardy–Orlicz type spaces appear naturally when studying the theory of nonlinear PDEs (cf. [24, 29, 31]) since many cancellation phenomena for Jacobians cannot be observed in the usual Hardy spaces $H^p(\mathbb{R}^n)$. For instance, let $f = (f_1, \ldots, f^n)$ in the Sobolev class $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and the Jacobians $J(x, f)dx = df^1 \wedge \cdots \wedge df^n$, then (see Theorem 10.2 of [31])

$$\mathcal{T}(J(x, f)) \in L^1(\mathbb{R}^n) + H^\Phi(\mathbb{R}^n)$$

where $\Phi(t) = t/\log(e + t)$ and $\mathcal{T}(f) = f \log |f|$, since $J(x, f) \in H^1(\mathbb{R}^n)$ (cf. [15]) and $\mathcal{T}$ is well defined on $H^1(\mathbb{R}^n)$. We refer readers to [30, 49] for this interesting nonlinear operator $\mathcal{T}$.

In this paper we want to allow generalized Hardy–Orlicz spaces related to generalized Orlicz functions that may vary in the spatial variables. More precisely the Orlicz function $\Phi(t)$ is replaced by a function $\varphi(x, t)$, called Musielak–Orlicz function (cf. [17, 45]). We then define Hardy spaces of Musielak–Orlicz type. Apart from interesting theoretical considerations,