A Jost–Pais-Type Reduction of Fredholm Determinants and Some Applications

Alan Carey, Fritz Gesztesy, Denis Potapov, Fedor Sukochev and Yuri Tomilov

Abstract. We study the analog of semi-separable integral kernels in $\mathcal{H}$ of the type

$$K(x, x') = \begin{cases} 
F_1(x)G_1(x'), & a < x' < x < b, \\
F_2(x)G_2(x'), & a < x < x' < b, 
\end{cases}$$

where $-\infty \leq a < b \leq \infty$, and for a.e. $x \in (a, b)$, $F_j(x) \in \mathcal{B}_2(\mathcal{H}_j, \mathcal{H})$ and $G_j(x) \in \mathcal{B}_2(\mathcal{H}, \mathcal{H}_j)$ such that $F_j(\cdot)$ and $G_j(\cdot)$ are uniformly measurable, and

$$\|F_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H}_j, \mathcal{H})} \in L^2((a, b)), \quad \|G_j(\cdot)\|_{\mathcal{B}_2(\mathcal{H}, \mathcal{H}_j)} \in L^2((a, b)), \quad j = 1, 2,$$

with $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1, 2$, complex, separable Hilbert spaces. Assuming that $K(\cdot, \cdot)$ generates a trace class operator $K$ in $L^2((a, b); \mathcal{H})$, we derive the analog of the Jost–Pais reduction theory that succeeds in proving that the Fredholm determinant $\text{det}_{L^2((a, b); \mathcal{H})}(I - \alpha K)$, $\alpha \in \mathbb{C}$, naturally reduces to appropriate Fredholm determinants in the Hilbert spaces $\mathcal{H}$ (and $\mathcal{H}_1 \oplus \mathcal{H}_2$). Explicit applications of this reduction theory to Schrödinger operators with suitable bounded operator-valued potentials are made. In addition, we provide an alternative approach to a fundamental trace formula first established by Pushnitski which leads to a Fredholm index computation of a certain model operator.

Mathematics Subject Classification (2010). Primary: 47B10, 47G10; Secondary: 34B27, 34L40.

Keywords. Fredholm determinants, semi-separable kernels, Jost functions, perturbation determinants.

1. Introduction

The principal topic in this paper concerns semi-separable integral operators and their associated Fredholm determinants. In a nutshell, suppose that $\mathcal{H}$ and $\mathcal{H}_j$, $j = 1, 2$, are complex, separable Hilbert spaces, that $-\infty \leq a < b \leq \infty$, and introduce the semi-separable integral kernel in $\mathcal{H}$,
\[ K(x, x') = \begin{cases} F_1(x)G_1(x'), & a < x' < x < b, \\ F_2(x)G_2(x'), & a < x < x' < b, \end{cases} \] (1)

where for a.e. \( x \in (a, b) \), \( F_j(x) \in B(\mathcal{H}_j, \mathcal{H}) \) and \( G_j(x) \in B(\mathcal{H}, \mathcal{H}_j) \) such that \( F_j(\cdot) \) and \( G_j(\cdot) \) are uniformly measurable (i.e., measurable with respect to the uniform operator topology), and

\[ \|F_j(\cdot)\|_{B_2(\mathcal{H}_j, \mathcal{H})} \in L^2((a, b)), \|G_j(\cdot)\|_{B_2(\mathcal{H}, \mathcal{H}_j)} \in L^2((a, b)), \quad j = 1, 2. \] (2)

Assuming that \( K(\cdot, \cdot) \) generates a trace class operator \( K \) in \( L^2((a, b); \mathcal{H}) \), we derive the analog of the Jost–Pais reduction theory that naturally reduces the Fredholm determinant \( \det_{L^2((a, b); \mathcal{H})}(I - \alpha K) \), \( \alpha \in \mathbb{C} \), to appropriate Fredholm determinants in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \oplus \mathcal{H}_2 \) as described in detail in Theorem 2.13. For instance, we will prove the remarkable Jost–Pais-type reduction of Fredholm determinants [29] (also [12, 34]),

\[ \det_{L^2((a, b); \mathcal{H})}(I - \alpha K) = \det_{\mathcal{H}_1} \left( I_{\mathcal{H}_1} - \alpha \int_a^b dx \, G_1(x) \hat{F}_1(x; \alpha) \right) \]
\[ = \det_{\mathcal{H}_2} \left( I_{\mathcal{H}_2} - \alpha \int_a^b dx \, G_2(x) \hat{F}_2(x; \alpha) \right), \] (3)

where \( \hat{F}_1(\cdot; \alpha) \) and \( \hat{F}_2(\cdot; \alpha) \) are defined via the Volterra integral equations

\[ \hat{F}_1(x; \alpha) = F_1(x) - \alpha \int_x^b dx' \, H(x, x') \hat{F}_1(x'; \alpha), \] (4)
\[ \hat{F}_2(x; \alpha) = F_2(x) + \alpha \int_a^x dx' \, H(x, x') \hat{F}_2(x'; \alpha) \] (5)

(cf. (36) for the definition of \( H(\cdot, \cdot) \)).

To illustrate the ubiquity of semi-separable integral operators it suffices to consider the special finite-dimensional case and note that the integral kernel of the resolvent of any ordinary differential and finite difference operator with matrix-valued coefficients, on arbitrary intervals on the real line, yields a Green’s matrix of the type (1), cf. [20, Sect. XIV.3]. (The same applies to certain classes of convolution operators, cf. [20, Sect. XIII.10].) In particular, Schrödinger, Dirac, Jacobi, and CMV operators of great relevance to mathematical physics, are prime candidates to which this circle of ideas applies. In these cases the determinant reduction formulas (3) lead to natural extensions of well-known results due to Jost–Pais [29]. We also note that Jost functions of the type (3) are intimately related to Evans functions, a fundamental tool in linear stability theory associated with classes of non-linear evolution equations. In the latter context we note the frequent necessity to consider non-self-adjoint operators as the result of a linearization process and stress that (infinite) determinants are ideally suited to analyze certain spectral properties of non-self-adjoint operators. Moreover, as shown in [15], suitable 2-modified Fredholm determinant extensions of this approach also apply to convolution integral operators, whose kernel is associated with a