Dual Toeplitz Operators on the Sphere
Via Spherical Isometries

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Abstract. We solve a characterization problem for dual Hardy-space Toeplitz operators on the unit sphere \( S_n \) in \( \mathbb{C}^n \) posed by Guediri (Acta Math Sin (English series) 29(9):1791–1808, 2013). Our proof relies on the observation that dual Toeplitz operators on the orthogonal complement \( H^2(S_n)^\perp \) of the Hardy space in \( L^2 \) can be viewed as Toeplitz operators with respect to a suitable spherical isometry. This correspondence also allows us to determine the commutator ideal of the dual Toeplitz \( C^* \)-algebra.

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1. Introduction

Let \( \mathbb{B}_n \) denote the open Euclidean unit ball in \( \mathbb{C}^n \), \( S_n = \partial \mathbb{B}_n \) its boundary and \( \sigma \) the normalized surface measure on \( S_n \). Given an element \( \varphi \in L^\infty(\sigma) \), the Toeplitz operator \( T_\varphi \) with symbol \( \varphi \) acting on the Hardy space

\[
H^2(\sigma) = \mathbb{C}[z_1, \ldots, z_n]_{\text{deg} 2} \subset L^2(\sigma)
\]

is defined as the compression of the multiplication operator \( M_\varphi : L^2(\sigma) \to L^2(\sigma) \), \( f \mapsto \varphi f \) onto \( H^2(\sigma) \), that is,

\[
T_\varphi : H^2(\sigma) \to H^2(\sigma), \quad f \mapsto P_{H^2(\sigma)} M_\varphi f.
\]

By definition a dual Toeplitz operator on \( H^2(\sigma)^\perp \) is an operator of the form

\[
S_\varphi : H^2(\sigma)^\perp \to H^2(\sigma)^\perp, \quad f \mapsto P_{H^2(\sigma)^\perp} M_\varphi f.
\]

In contrast to the case of ordinary Toeplitz operators which have been introduced more than half a century ago, the investigation of the dual case has just begun—at least in the Hardy space situation (see the recent work [8] of Guediri). On the Bergman space of the unit disc \( \mathbb{D} \), these operators have been introduced and studied in detail by Stroethoff and Zheng [11]. A corresponding theory on the Hardy space \( H^2(\partial \mathbb{D}) \) over the unit disc contains nothing
new, since there is a natural isomorphism $H^2(\partial \mathbb{D}) \cong H^2(\partial \mathbb{D})^\perp$, under which Toeplitz and dual Toeplitz operators are equivalent. But in complex dimension $n > 1$, new phenomena occur (see Proposition 1.3 and the subsequent remarks in [8], or [3] where it is shown that $T_z \in B(H^2(\sigma))^n$ is not even quasi-similar to $S_z$ for $n > 1$).

From now on, we suppose that $n > 1$. Given $\varphi \in L^\infty(\sigma)$, the orthogonal decomposition $L^2(\sigma) = H^2(\sigma) \oplus H^2(\sigma)^\perp$ yields a representation of the multiplication operator $M_\varphi \in B(L^2(\sigma))$ as an operator-matrix of the form

$$M_\varphi = \begin{pmatrix} T_\varphi & H^\perp_\varphi \\ H_\varphi & S_\varphi \end{pmatrix},$$

where $H_\varphi : H^2(\sigma) \to H^2(\sigma)^\perp, f \mapsto P_{H^2(\sigma)^\perp} M_\varphi f$, is the so-called Hankel operator with symbol $\varphi$. Various algebraic relations connecting the operators $T_\varphi, S_\varphi$ and $H_\varphi$ (see the equations (2.1) and Lemma 2.1 in [8]) result from this representation. Moreover, by $M_\varphi^* = M_\varphi$ it follows immediately that $S_\varphi^* = S_\varphi$.

The aim of this note is to demonstrate that the theory of dual Toeplitz operators fits into the more general context of Toeplitz operators with respect to spherical isometries. To be more specific, the tuple $T = (S_{z_1}, \ldots, S_{z_n}) \in B(H^2(\sigma)^\perp)$ is a spherical isometry, and the dual Toeplitz operators on $H^2(\sigma)^\perp$ are precisely the associated $T$-Toeplitz operators (see Proposition 2.1). This correspondence immediately leads to some known results on dual Toeplitz operators and allows us to establish a short exact sequence for the dual Toeplitz $\mathcal{C}^*$-algebra $\mathcal{C}^*(S_f : f \in C(S_n))$ (see Sect. 3). As another application we solve a problem posed by Guediri ([8, Remark 3.3]) concerning the characterization of dual Toeplitz operators (see Sect. 4). The following section contains the necessary background on spherical isometries and their associated Toeplitz operators.

**2. Spherical Isometries and Dual Toeplitz Operators**

Let $H$ be a separable complex Hilbert space. A spherical isometry $T \in B(H)^n$ is a commuting tuple of operators satisfying the algebraic condition

$$\sum_{i=1}^n T_i^* T_i = 1_H$$

which is modelled after the relation $\sum_{i=1}^n |z_i|^2 = 1$ describing the unit sphere in $\mathbb{C}^n$. Perhaps the most important example of a spherical isometry is given by the Toeplitz tuple $T_z = (T_{z_1}, \ldots, T_{z_n}) \in B(H^2(\sigma))^n$ on the Hardy space of the sphere. The study of general spherical isometries has been initiated by Athavale in 1990 who proved that they are subnormal and that their minimal normal extension is again a spherical isometry (see [2]). Note that a normal tuple is spherical isometric if and only if its spectrum is contained in the unit sphere. It is called a spherical unitary in this case.

Inspired by the prototypical example $T_z$, Prunaru developed a general theory of Toeplitz operators for spherical isometries which—in many respects—parallels the classical theory of Hardy-space Toeplitz operators on