Existence of Weak Solutions for the Three-Dimensional Motion of an Elastic Structure in an Incompressible Fluid

Muriel Boulakia

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Abstract. We study here the three-dimensional motion of an elastic structure immersed in an incompressible viscous fluid. The structure and the fluid are contained in a fixed bounded connected set $\Omega$. We show the existence of a weak solution for regularized elastic deformations as long as elastic deformations are not too important (in order to avoid interpenetration and preserve orientation on the structure) and no collisions between the structure and the boundary occur. As the structure moves freely in the fluid, it seems natural (and it corresponds to many physical applications) to consider that its rigid motion (translation and rotation) may be large. The existence result presented here has been announced in [4]. Some improvements have been provided on the model: the model considered in [4] is a simplified model where the structure motion is modelled by decoupled and linear equations for the translation, the rotation and the purely elastic displacement. In what follows, we consider on the structure a model which represents the motion of a structure with large rigid displacements and small elastic perturbations. This model, introduced by [15] for a structure alone, leads to coupled and nonlinear equations for the translation, the rotation and the elastic displacement.

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1. Introduction and equations of the motion

On the elastic structure, we have a rigid motion combined with an elastic motion with small deformations. More precisely, the lagrangian flow $X_S$ is defined by

$$X_S(t,0,y) = a(t) + Q(t)(y - g_0) + Q(t)\xi(t,y), \forall y \in \Omega_S(0), \forall t \in [0,T],$$  \hspace{1cm} (1.1)

where $\Omega_S(0)$ is an open regular set which represents the initial domain occupied by the structure, $g_0$ is the center of mass of the solid at time $t = 0$, $a$ the translation of the structure, $Q \in SO_3(\mathbb{R})$ the rotation of the structure and $\xi$ the elastic deformation of the structure. The vector $X_S(t,0,y)$ gives the position at time $t$ of
the particle located in $y$ at initial time. We suppose that

$$\int_{\Omega_S(0)} \rho_S^0(y) \xi(t, y) \, dy = 0, \quad \int_{\Omega_S(0)} \rho_S^0(y) \xi(t, y) \wedge (y - g_0) \, dy = 0, \quad (1.2)$$

where $\rho_S$ is the density of the solid at time $t = 0$ which satisfies

$$0 < M_1 \leq \rho_S^0 \leq M_2 \text{ on } \Omega_S(0),$$

where $M_1$ and $M_2$ are two positive constants. These conditions mean that the elastic motion is orthogonal to the infinitesimal translations and rotations.

We also define the lagrangian velocity $U_S$ by

$$U_S(t, y) = \partial_t X_S(t, 0, y), \quad \forall y \in \Omega_S(0), \quad \forall t \in [0, T]$$

$$= \dot{a}(t) + \omega(t) \wedge Q(t)(y - g_0 + \xi(t, y)) + Q(t)\partial_3 \xi(t, y),$$

where the rotation velocity vector $\omega$ is defined in $\mathbb{R}^3$ by

$$\forall t \in [0, T], \quad \forall x \in \mathbb{R}^3, \quad Q(t)Q(t)^{-1}x = \omega(t) \wedge x.$$ The lagrangian flow defines at each time the structure domain and the fluid domain. Let us denote

$$\Omega_S(t) = X_S(t, 0, \Omega_S(0)) \quad \text{and} \quad \Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)},$$

which represent respectively the domain occupied by the structure at time $t$ and the domain occupied by the fluid at time $t$. Moreover, we suppose that the elastic deformations are small enough to get an invertible flow from $\Omega_S(0)$ onto $\Omega_S(t)$ (this hypothesis of smallness will be satisfied by our solution). Thus, we can define the eulerian velocity $u_S$ by

$$u_S(t, x) = \partial_t X_S(t, 0, X_S(0, t, x)), \quad \forall x \in \Omega_S(t), \quad \forall t \in [0, T], \quad (1.3)$$

where $X_S(0, t, \cdot)$ denotes the inverse of $X_S(t, 0, \cdot)$. By this way, $u_S$ can be expressed with respect to $a$, $Q$ and $\xi$:

$$\forall t \in [0, T], \quad \forall x \in \Omega_S(t), \quad u_S(t, x) = \dot{a}(t) + \omega(t) \wedge (x - a(t)) + Q(t)\partial_3 \xi(t, X_S(0, t, x)).$$

As the flow is invertible, we can also define

$$X_S(t, s, x) = X_S(t, 0, X_S(0, s, x)), \quad \forall x \in \Omega_S(s), \quad \forall t, s \in [0, T].$$

The vector $X_S(t, s, x)$ gives the position at time $t$ of the particle located in $x$ at time $s$.

On the fluid domain, we have an eulerian point of view: let $u_F$ be the eulerian velocity of the fluid. At last, we denote $u$ the global eulerian velocity on $\Omega$ and $X$ the associated lagrangian flow.

The unknowns of our problem are $a$, $Q$ or $\omega$ the corresponding rotation velocity vector, $\xi$, $u_F$, $\rho_F$ the density of the fluid and $p$ the pressure of the fluid.

The motion of the fluid is described by the incompressible viscous Navier–Stokes equations and the evolution of the fluid density is governed by the mass conservation law:

$$\rho_F (\partial_t u_F + (u_F \cdot \nabla)u_F) - \text{div} \sigma_F = 0 \text{ on } \Omega_F(t), \quad (1.4)$$