Propagation of Density-Oscillations in Solutions to the Barotropic Compressible Navier–Stokes System

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Abstract. Considering a bounded sequence of weak solutions to the compressible Navier–Stokes system, we introduce Young measures as in [12] in order to describe a “homogenized system” satisfied in the limit. We then study the Cauchy problem associated to this “homogenized system” when Young measures are convex combinations of Dirac measures.


Keywords. Compressible Navier–Stokes system, homogenization, strong solution.

1. Introduction

One model currently used to describe the motion of a barotropic viscous compressible fluid reads:

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \text{div}(\mathbf{u}), \\
p &= p(\rho) = a \rho^\gamma,
\end{align*}
\]

in \((0, T) \times \Omega. \quad (1)\]

Here \(T\) is a strictly positive time and \(\Omega\) is a smooth open bounded domain in \(\mathbb{R}^3\). The notation \(\rho := \rho(t, \mathbf{x}) \in \mathbb{R}^+\) (resp. \(\mathbf{u} := \mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^3\)) stands for the density (resp. velocity) of the fluid inside the cavity. In order to obtain a well-posed problem, we complement with initial and boundary conditions:

\[
\begin{align*}
\mathbf{u}\big|_{t=0} &= \mathbf{u}^0, \\
\mathbf{u}\big|_{\partial \Omega} &= 0, \\
(\rho \mathbf{u})\big|_{t=0} &= \mathbf{m}^0.
\end{align*}
\]

Initial condition on \(\rho \mathbf{u}\) may be stated in terms of \(\mathbf{u}\) instead. Throughout the paper, we refer to the full system \((1, 2)\) as \((\text{CNS})\). Constants \((a, \mu, \lambda)\) satisfy:

\[a > 0, \quad \mu > 0, \quad \text{and} \quad 3\lambda + 2\mu \geq 0.\]
(CNS) shares hyperbolic (equation (1a)) and parabolic (equation (1b)) properties. In particular, the density satisfying an hyperbolic equation, initial oscillations may persist in time. We derive here a system in order to describe the propagation of these oscillations and present a study of the Cauchy problem for this system in a particular case.

1.1. Notation

In the whole paper, we denote vector quantities by bold faces. We denote derivative by subscripts. The operator div, ∇ and Δ are the classical divergence, gradient and Laplace operator. When taking the divergence of a matrix function, we mean the vector of the divergences of each line, considered as a vector function. The cylinder \((0, T) \times Ω\) is often denoted \(Q_T\). When considering a vector-function \(r = (r_1, \ldots, r_m)\) defined over a domain \(Ω\), we shall use:

\[
 F := \sup_{i=1,\ldots,m} \sup_{y \in Ω} r_i(y), \quad F^* := \inf_{i=1,\ldots,m} \inf_{y \in Ω} r_i(y).
\]

Concerning functional spaces, most of notation is classical. We refer to smooth functions with compact support in \(Q_T\) as \(D(Q_T)\) and to its dual space as \(D'(Q_T)\). For the sake of conciseness, we do not precise in functional space names when they are concerned with real-valued or (finite-dimensional) vector-valued functions. We denote Lebesgue and Sobolev spaces by \(L^p(Ω)\) and \(W^{m,p}(Ω)\). When \(p = 2\), we also use the shorthand \(H^m(Ω)\). On these spaces, the norms are denoted respectively, \(∥ · ∥_p\), \(∥ · ∥_{m,p}\) and \(∥ · ∥_m\). The notation \(H^m_0(Ω)\) stands for the closure of \(D(Ω)\) inside \(H^m(Ω)\).

Since we deal with evolutionary partial differential equations, functions may depend on time. Hence, we denote by \(L^q(0, T; L^p(Ω))\) (resp. \(L^q(0, T; W^{m,p}(Ω))\) and \(L^q(0, T; H^m(Ω))\)) the set of \(L^q(0, T)\) functions with values in \(L^p(Ω)\) (resp. in \(W^{m,p}(Ω)\) and \(H^m(Ω)\)). In the two latter spaces, norms are denoted by \(∥ · ∥_{q,m,p}\) and \(∥ · ∥_{q,m}\). Obviously, the norm in the first space is \(∥ · ∥_{q,0,p}\).

Finally, we introduce:

\[
 B_p(M) = \{ f \in L^p(Ω) \text{ such that } |f|_p \leq M \},
\]

and \(w\) (resp. \(w^*\)) refers to the weak (resp. weak-star) topology. Throughout the whole paper, we will make repeated use of the embeddings

\[
 H^2(Ω) \subset L^∞(Ω), \quad H^1(Ω) \subset L^p(Ω), \quad \forall p \leq 6.
\]

In the first part of our study we also introduce the duality bracket \(⟨ · , · ⟩\) between measures and bounded continuous functions. In these brackets, the right-hand side is a continuous function. It is denoted with straight letters. In particular, \(ι\) is the identity on \(R\) and \(p : x \mapsto ax^γ\).

Finally, within a system, we label equations with letters. For example, (1b), as we used previously is the second equation inside system (1).