Conclaves of planes in $\text{PG}(4, 2)$ and certain planes external to the Grassmannian $\mathcal{G}_{1,4,2} \subset \text{PG}(9, 2)$

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Abstract. We show that in $\text{PG}(4, 2)$ there exist octets $P_8 = \{\pi_1, \ldots, \pi_8\}$ of planes such that the 28 intersections $\pi_i \cap \pi_j$ are distinct points. Such conclaves (see [6]) $P_8$ of planes in $\text{PG}(4, 2)$ are shown to be in bijective correspondence with those planes $P$ in $\text{PG}(9, 2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ and which belong to the orbit $\text{orb}(2\gamma)$ (see [4]). The fact that, under the action of $\text{GL}(5, 2)$, the stabilizer groups $G_{P_8}$ and $G_P$ both have the structure $2^3 : (7 : 3)$ is thus illuminated. Starting out from a regulus-free partial spread $S_8$ in $\text{PG}(4, 2)$ we also give a construction of a con clave of planes $P \in \text{orb}(2\gamma) \subset \text{PG}(9, 2)$.

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1. Introduction: notation and background results

In this section we summarize some relevant background material in the three areas (i) partial spreads of lines in $\text{PG}(4, 2)$; (ii) planes in $\text{PG}(9, 2)$ which are external to the Grassmannian $\mathcal{G}_{1,4,2}$ of lines of $\text{PG}(4, 2)$; (iii) conclaves of planes in $\text{PG}(n, 2)$.

First a few words concerning our notation. Since we work throughout over $\text{GF}(2)$, we will identify the nonzero elements of a vector space $V_{n+1}$ with the points of the associated projective space $\text{PG}(n+1) = \text{PG}(n, 2)$. Consequently we identify $\text{GL}(V_{n+1}) = \text{GL}(n+1, 2)$ with the group $\text{PGL}(n+1, 2)$ of collineations of $\text{PG}(n, 2)$. Similarly nonzero elements of the dual vector space $V^*_n$ will be identified with the points of the dual projective space $\text{PG}(n, 2)^*$. We use $\langle v_1, \ldots, v_r \rangle$ for the vector subspace of $V_{n+1}$ which is spanned by vectors $v_1, \ldots, v_r \in V_{n+1}$, and $\langle v_1, \ldots, v_r \rangle$ for the projective subspace of $\text{PG}(n, 2)$ generated by points $v_1, \ldots, v_r \in \text{PG}(n, 2)$. Recall that the annihilator $U^O := \{ f \in V^*_n \mid f(u) = 0, \text{ for all } u \in U \}$ of any subset $U \subset V_{n+1}$ is always a subspace of $V^*_n$; moreover if $\dim U = r$ then $\dim U^O = n - r$. We use the same notation also for projective subspaces. Thus if $\alpha$ is a plane in $\text{PG}(4, 2)$ then $\alpha^O$ is a line in $\text{PG}(4, 2)^*$.

We will be particularly concerned with a vector space $V_5 = V(5, 2)$, and its projective space $\text{PG}(4, 2) = \mathbb{P}V_5$, along with the concomitant space $V_{10} := \wedge^2 V_5$ of bivectors. In
Section 3.2 we will view the dual $V_{10}^*$ of $V_{10}$ as the space $V_{10}^* := \wedge^3 V_5$ of trivectors, by means of the natural nondegenerate bilinear pairing $[\cdot, \cdot]$ of $\wedge^3 V_5$ with $\wedge^2 V_5$ defined by
\[ t \wedge b = [t, b] \varepsilon, \quad t \in \wedge^3 V_5, \quad b \in \wedge^2 V_5. \] (1.1)

Here $\varepsilon$ is the (unique!) basis vector for the 1-dimensional space $\wedge^5 V_5$. Each $A \in \text{GL}(5, 2)$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\text{GL}(V_{10})$ whose effect on the decomposable bivectors $u \wedge v \in V_{10}$ is $T_A(u \wedge v) = Au \wedge Av$, $A \in \text{GL}(5, 2)$. Similarly we put $\hat{T}_A = \wedge^3 A \in \text{GL}(V_{10}^*)$. Since $(\wedge^5 A)e = e$ for all $A \in \text{GL}(5, 2)$, note the invariance property
\[ \hat{T}_A T_A b = t \wedge b, \quad t \in \wedge^3 V_5, \quad b \in \wedge^2 V_5, \quad \text{for all } A \in \text{GL}(5, 2). \] (1.2)

Thus $\hat{T}_A$ is the contragredient of $T_A : [\hat{T}_A t, T_A b] = [t, b]$.

If X is an object of some $\text{GL}(5, 2)$-space then $G_X < \text{GL}(5, 2)$ denotes its stabilizer group. In particular if X is an object in $\wedge^2 V_5$ its stabilizer is $G_X = \{ A \mid A \in \text{GL}(5, 2), T_A(X) = X \}$. Under the action $T$ of $\text{GL}(5, 2)$ the projective space $\text{PG}(9, 2) = \mathbb{P}(\wedge^2 V_5)$ is the union $Rk_2 \cup Rk_4$ of two $\text{GL}(5, 2)$-orbits, consisting of those bivectors having rank 2 and rank 4, respectively. The Plücker map $< u, v > \mapsto < u \wedge v >$ sends the 2-spaces of $V_5$ to those 1-spaces of $\wedge^2 V_5$ which are spanned by decomposable bivectors. Projectively, the lines of $\text{PG}(4, 2)$ are mapped onto the points of the orbit $Rk_2$, the latter, being the Grassmannian $G_{1,4,2} \subset \text{PG}(9, 2)$ of lines of $\text{PG}(4, 2)$, having length 155. Consequently $|Rk_4| = 1023 - 155 = 868$. Throughout this paper the images in $G_{1,4,2} \subset \text{PG}(9, 2)$ of lines $\lambda, \mu$ in $\text{PG}(4, 2)$ will be denoted $l, m$. Similarly, under the action $\hat{T}$ of $\text{GL}(5, 2)$, the projective space $\text{PG}(9, 2)^* = \mathbb{P}(\wedge^3 V_5)$ is the union of two orbits of lengths 155 and 868. Using the Plücker map $< u, v, w > \mapsto < u \wedge v \wedge w >$, the 155 planes of $\text{PG}(4, 2)$ are mapped onto the 155 points of the Grassmannian $G_{2,4,2} \subset \text{PG}(9, 2)^*$. The image in $G_{2,4,2}$ of a plane $\pi$ in $\text{PG}(4, 2)$ is denoted $p$.

1.1. Partial spreads in $\text{PG}(4, 2)$

A partial spread $S_r$ in $\text{PG}(4, 2)$ of size $r (> 0)$ is a set $\{ \mu_1, \ldots, \mu_r \}$ of $r$ pairwise disjoint lines. Such partial spreads have recently been completely classified: see [2, Table B.2]. Under the action of $\text{GL}(5, 2)$ they fall into 64 distinct classes (GL(5, 2)-orbits). In particular the partial spreads $S_6$ of size 9 (the maximum size possible) consist of four classes, IXa.1, IXa.2, IXa.3 and IXa.4, and the partial spreads $S_8$ of size 8 consist of nine classes, VIIIa.1, VIIIb.1 and VIIIc.1-VIIIc.7. We will be especially interested in the partial spreads $S_8$ of class VIIIa.1; such a partial spread has a relatively large stabilizer group $G_{S_8}$, of order 168, which acts transitively on $S_8$. (The stabilizer groups for the other eight classes of $S_8$ are all of order $\leq 6$, and are not transitive.) In the next theorem we summarize a few