On mappings of $Q^d$ to $Q^d$ that preserve distances 1 and $\sqrt{2}$ and the Beckman-Quarles Theorem

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Abstract. Benz proved that every mapping $f : Q^d \to Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$. We prove that every mapping $f : Q^d \to Q^d$ that preserves the distances 1 and $\sqrt{2}$ is an isometry, provided $d \geq 5$.

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Let $F$ be any subfield of the reals $\mathbb{R}$ and let $F^d$ denote the subspace of $\mathbb{R}^d$, in which all the coordinates belong to $F$. A mapping $f : F^d \to F^d$ is called a $\rho$-distance preserving mapping, for a real $\rho$, if $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$. The Beckman-Quarles Theorem [1, 3, 9, 10, 15] states that every unit-distance preserving mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is an isometry, provided $d \geq 2$. A few papers [7, 8, 11, 14, 15] treat the rational analogues of this theorem, i.e., treating, for some values of $d$, the property “Every unit-distance preserving mapping $f : Q^d \to Q^d$ is an isometry”. This statement is false for $d = 2, 3,$ and 4 [4], it is true for all even $d \geq 6$ [7] and it is true for all odd $d$ of the form $2n^2 - 1, n \geq 3$ [15]. One tool used in these results is Benz’s theorem ([4], see also [8]), which states that every mapping $f : Q^d \to Q^d$ that preserves the distances 1 and 2 is an isometry, provided $d \geq 5$. The purpose of this note is to give a Beckman-Quarles type theorem and to extend Benzs’s result, as follows.

THEOREM 1. If a mapping $f : Q^d \to Q^d$ preserves the distance $\sqrt{2}$, then $f$ is an isometry, provided $d$ is even and $d \geq 6$, or $d = 4k^2 - 1$, for $k \geq 2$.

THEOREM 2. If a mapping $f : Q^d \to Q^d$ preserves the distance $\sqrt{2}$, then $f$ preserves the distance 2, provided $d \geq 5$.

THEOREM 3. If a mapping $f : Q^d \to Q^d$ preserves the distances 1 and $\sqrt{2}$, then $f$ is an isometry, provided $d \geq 5$.

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Let $F(d, \rho)$ denote the graph whose vertices are the points of $P^d$, and where two vertices $x$ and $y$ are connected by an edge if, and only if, $\|x - y\| = \rho$. Let $\omega(G)$ denote the clique number of the graph $G$.

We need the following Lemmas.

**LEMMA 1.** The value of $\omega(Q(d, \sqrt{2}))$ is given by:

$$\omega(Q(d, \sqrt{2})) = \begin{cases} d + 1 & \text{if } d + 1 \text{ is a square} \\ d & \text{otherwise} \end{cases}$$

**Proof.** Obviously $\omega(Q(d, \sqrt{2})) \leq d + 1$, since $\omega(F(d, \rho)) \leq d + 1$ holds for all $F$ and all $\rho$. Let $e_1, e_2, \ldots, e_d$ be the rows of the unit matrix $I$ of order $d$; they form a set of $d$ points in $Q^d$ of mutual distance $\sqrt{2}$. It follows that $\omega(Q(d, \sqrt{2})) \geq d$. The only two possibilities for an additional point in $Q^d$ to be at distance $\sqrt{2}$ to these $d$ points is the point $X = (x, x, \ldots, x)$ for $x = [1 + \sqrt{2d+1}]$. Therefore, if $\omega(Q(d, \sqrt{2})) = d + 1$, then $d + 1$ is a square. The last implication follows also from the Euler-Kelly-Menger formula, which will be mentioned at the end of the paper. This completes the proof of Lemma 1.

For an even $d$, let $h_d : Q^d \to Q^d$ be the mapping given by the block-matrix $H$ of size $d$ by $d$, as given by the following form:

$$H = \begin{pmatrix} T & 0 & \cdots & 0 \\ 0 & T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T \end{pmatrix}, \text{ where } T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$  

The mapping $h_d$ is a rational mapping that shrinks $Q^d$ by the factor $\frac{1}{\sqrt{2}}$, while the mapping $h_d^{-1}$ is a rational mapping that expands $Q^d$ by the factor $\sqrt{2}$. $\square$

**Proof of Theorem 1.** Let $f : Q^d \to Q^d$ be a $\sqrt{2}$-distance preserving mapping. If $d$ is even and $d \geq 6$, then the mapping $h_d \circ f \circ h_d^{-1} : Q^d \to Q^d$ is a unit-distance preserving mapping, hence it is a congruence by [7]. Therefore the mapping $h_d \circ f \circ h_d^{-1} \circ h_d = f$ is also an isometry. If $d = 4k^2 - 1$, for $k \geq 2$, then $d + 1 = (2k)^2$, hence the space $Q^d$ contains, by Lemma 1, a $d$-simplex of edge length $\sqrt{2}$. By using arguments, similar to those that appeared in the proof of the Beckman-Quarles Theorem, given in [15], it follows that the mapping $f$, which preserves the distances $\sqrt{2}$, is an isometry. This completes the proof of Theorem 1.

**COROLLARY 1** (see also Corollary 2 in [7]). If $t$ is the sum of two squares of rational numbers, then every mapping $f : Q^d \to Q^d$ that preserves the distances $\sqrt{t}$ is an isometry, provided $d$ is even and $d \geq 6$. 