On Polar Spaces of Infinite Rank

Antonio Pasini

Abstract. Many properties of polar spaces of finite rank fail to hold in polar spaces of infinite rank. For instance, in a polar space of infinite rank it can happen that maximal singular subspaces have different dimensions; every polar space of infinite rank contains singular subspaces that cannot be obtained as intersections of any family of maximal singular subspaces, whereas in a polar space of finite rank every singular subspace is the intersection of a finite number of maximal singular subspaces. In this paper we shall examine peculiar properties of polar spaces of infinite rank, to test how far they can drug us from the familiar world of polar spaces of finite rank.

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1. Introduction

We firstly recall some basics on partial linear spaces and polar spaces. Let $S = (P, L)$ be a partial linear space, where $P$ is the point-set, $L$ is the line-set and lines are regarded as subsets of $P$. We assume that every line of $S$ contains at least two points, but we allow $L$ to be empty. We write $x \perp y$ to say that two points $x, y \in P$ are collinear and denote by $x^\perp$ the set of points of $S$ collinear with $x$, including $x$ among them. For a subset $X \subseteq P$, we put $X^\perp := \cap_{x \in X} x^\perp$.

A subset $X \subseteq P$ is called a subspace of $S$ if it contains every line that meets $X$ in at least two points. Intersections of arbitrary families of subspaces are subspaces. The subspace $\langle X \rangle$ of $S$ spanned by a subset $X \subseteq P$ is the intersection of all subspaces of $S$ containing $X$.

A subspace $X$ of $S$ is said to be singular if $X \subseteq X^\perp$, namely any two points of $X$ are collinear. The union of a chain of singular subspaces is a singular subspace. So,
by Zorn’s Lemma, every singular subspace of \( S \) is contained in a maximal singular subspace. Denoted by \( \Sigma \) the family of all chains of singular subspaces of \( S \), we put
\[
\text{rank}(S) := \min(n \mid n \geq |C| - 1 \text{ for all } C \in \Sigma)
\]
and we call \( \text{rank}(S) \) the rank of \( S \). Needless to say, in the above definition the letter \( n \) stands for a (possibly infinite) cardinal number. We will follow the same convention throughout this paper, always keeping bold-type letters as \( n, m, d, \ldots \) for possibly infinite cardinal numbers.

The rank \( \text{rank}(X) \) of a subspace \( X \) of \( S \) is the rank of the partial linear space induced by \( S \) on \( X \), which we also denote by the letter \( X \). Singular subspaces of rank 3 are called planes. Clearly,
\[
\text{rank}(S) = \min \{ n \mid n \geq \text{rank}(M) \text{ for all maximal singular subspaces } M \text{ of } S \}.
\]

Let \( X \) be a singular subspace of \( S \) and suppose that \( S \) induces a projective space on \( X \). Then \( \text{rank}(X) = 1 + \dim(X) \), with the usual convention that \( 1 + n = n \) if \( n \) is an infinite cardinal number.

A proper subspace of \( S \) is called a hyperplane if it is a subspace of \( S \) and meets every line of \( S \) non-trivially. Following Buekenhout and Shult [5] (also Beuekenhout and Cohen [2]), we say that \( S \) is a polar space if \( x \perp \) is a hyperplane of \( S \), for every point \( x \in P \setminus P \perp \).

For the rest of this introduction \( S \) is a polar space. Then \( X \perp \) is a subspace of \( S \) for every subset \( X \subseteq P \). In particular, \( P \perp \) is a (possibly empty) singular subspace of \( S \), called the radical of \( S \). In fact, \( P \perp \) is the intersection of all maximal singular subspaces of \( S \). If \( P \perp = \emptyset \), then the polar space \( S \) is said to be non-degenerate. If all lines of \( S \) contain at least three points, then \( S \) is said to be irreducible. Throughout this paper, irreducible non-degenerate polar spaces will be called ordinary.

Non-ordinary polar spaces can be reduced to ordinary polar spaces by means of two basic constructions, namely direct sums and quotients. Indeed, reducible polar spaces can be described as direct sums of irreducible polar spaces (Buekenhout and Sprague [6]), while the quotient of a degenerate polar space by its radical is a non-degenerate polar space. We are not going to discuss direct sums here. We refer the interested reader to [2, Chapter 9] or [6] for them. We shall briefly discuss quotients instead.

For a non-maximal singular subspace \( X \) of \( S \) and a subspace \( X \subseteq Y \subseteq X \perp \), the quotient \( Y/X \) of \( Y \) over \( X \) is the point-line geometry with point-set \( P_{X}^{Y} \) and line-set \( L_{X}^{Y} \) defined as follows
\[
P_{X}^{Y} = \left\{ \langle \{x\} \cup X \rangle \mid x \in Y \setminus X \right\},
\]
\[
L_{X}^{Y} = \left\{ \langle \{x, y\} \rangle \mid x, y \in Y \setminus X, \langle \{x\} \cup X \rangle \neq \langle \{y\} \cup X \rangle \right\},
\]
and containment as the incidence relation. When \( Y = X \) we put \( Y/X = \emptyset \), by convention.