Centroaffine space curves with constant curvatures and homogeneous surfaces

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Abstract. We study the geometric characters of a centroaffine space curve with vanishing centroaffine curvatures, and classify the centroaffine space curves with constant centroaffine curvatures, which are centroaffine homogeneous curves in \( \mathbb{R}^3 \). Moreover, we can find a centroaffine homogeneous surface on which such a space curve lies.

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1. Introduction

The fundamental theorem of curves in centroaffine geometry is obtained by Gardner and Wilkens [2]. In that paper they used the classical Cartan approach to moving frames in order to find the formulation of the local rigidity theorem for curves that is amenable to direct application to problems in control theory.

In Euclidean differential geometry, a curve in \( \mathbb{E}^3 \) is called rectifying if the position vector lies in its rectifying plane at each point [1]. We define the centroaffine rectifying curve by using the same geometric description and obtain the centroaffine second curvature of a curve vanishes if and only if it is centroaffine rectifying.

As it is known, a centroaffine curve in \( \mathbb{R}^2 \) with constant centroaffine curvature can be represented by some exponential functions of the centroaffine arc-length parameter [5]. In this paper, we study the centroaffine space curves with constant curvatures and give the following classification theorem:
Theorem 1.1. (Main theorem) Any nondegenerate centroaffine space curve with constant centroaffine curvatures and signature $-1$ is centroaffinely equivalent to one of the following curves:

1. $\varphi(s) = \{se^{-s}, e^{-s}, s^2e^{-s} + e^{-s}\}$, if $A^2 + B^2 = 0$,
2. $\varphi(s) = \{e^s, e^s, s^2e^s\}$, if $A^2 + B^2 \neq 0$ and $\Delta = 0$,
3. $\varphi(s) = e^{2s/3}\{e^{\rho_1} s \sin(\rho_2), e^{\rho_1} s \cos(\rho_2), e^{-2\rho_1}\}$, if $A^2 + B^2 \neq 0$ and $\Delta > 0$,
4. $\varphi(s) = e^{2s/3}\{e^{-2\rho_1}, e^{(\sigma_1 + \sigma_2)s}, e^{(\sigma_1 - \sigma_2)s}\}$, if $A^2 + B^2 \neq 0$ and $\Delta < 0$,

where $\kappa_2$ is the centroaffine second curvature, $A, B, \Delta$ and $\varsigma_i, \rho_i, \sigma_i (i = 1, 2)$ are constants with respect to centroaffine curvatures given by (3.2)~(3.6).

Moreover, we can express each curve as the orbit of a one parameter subgroup of $GL(3; \mathbb{R})$ as in Theorem 4.2. In particular, each curve of type (2), (3) and (4) lies on a certain nondegenerate centroaffine homogeneous surface which is determined by the centroaffine curvatures of the curve. This surface gives rise to a centroaffine minimal surface if the centroaffine second curvature of the curve vanishes. Furthermore, the group of the curve and the one of the surface satisfy subgroup and overgroup relation. However, we are not able to find any centroaffine homogeneous surface for the curve of type (1) such that the groups of the curve and the surface satisfy this relation.

2. Basic notions of centroaffine space curves

Definition 2.1. Let $I = (a, b)$ be an interval in $\mathbb{R}$, $\varphi : I \to \mathbb{R}^3$ a $C^\infty$ map. $\varphi(t)$ is called a centroaffine curve in $\mathbb{R}^3$ if $[\varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t)] \neq 0$, for all $t \in I$, where $[\ , \ ]$ is the standard volume form of $\mathbb{R}^3$, that is, the determinant.

Definition 2.2. A centroaffine curve $\varphi$ is called nondegenerate if for all $t \in I, [\dot{\varphi}(t), \ddot{\varphi}(t), \dddot{\varphi}(t)] \neq 0$.

Definition 2.3. A nondegenerate centroaffine curve is said to be parameterized by centroaffine arc-length parameter if $\varepsilon(s) := \frac{[\varphi(s), \dot{\varphi}(s), \ddot{\varphi}(s)]}{[\varphi(s), \dot{\varphi}(s), \dddot{\varphi}(s)]} = \pm 1$, for all $s \in I$, where $\varepsilon$ is called the signature of $\varphi$, $s$ is called the centroaffine arc-length parameter of $\varphi$.

Remark 2.4. Any nondegenerate centroaffine curve has a reparametrization by centroaffine arc-length parameter.

In the following we use $"\cdot"$ as the differential operator with respect to the general parameter and $"\cdot\cdot\cdot"$ as the one with respect to the centroaffine arc-length parameter.

Definition 2.5. For a nondegenerate centroaffine curve parameterized by centroaffine arc-length parameter, we define the centroaffine first curvature by

$$\kappa_1(s) := \frac{[\varphi(s), \varphi''(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]},$$