On $\alpha$-conformal equivalence of statistical submanifolds

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Abstract. In this paper, we show a procedure to realize a statistical manifold, which is $\alpha$-conformally equivalent to a manifold with an $\alpha$-transitively flat connection, as a statistical submanifold.

Mathematics Subject Classification (2000): 53A15.
Key words: $\alpha$-connections, $\alpha$-transitively flat connections, $\alpha$-conformal equivalence.

1. Introduction

Statistical manifolds are studied in terms of information geometry. The theory of $\alpha$-connections of statistical manifolds plays an important role especially on statistical inference. In addition, considering conformal transformation into $\alpha$-connections, Okamoto, Amari and Takeuchi obtain asymptotic theory of sequential estimation [3]. Kurose defined $\alpha$-conformal equivalence and $\alpha$-conformal flatness of statistical manifolds [2]. In our previous paper, we gave an example for a 1-conformally flat statistical submanifold of a flat statistical manifold, using a Hessian domain [4] [5]. In this paper, we show a procedure to realize a statistical manifold, which is $\alpha$-conformally equivalent to a manifold with an $\alpha$-transitively flat connection, as a statistical submanifold. An $\alpha$-transitively flat connection is one of $\alpha$-connections.

2. $\alpha$-transitively flat connections on statistical manifolds

For a torsion-free affine connection $\nabla$ and a pseudo-Riemannian metric $h$ on a manifold $N$, the triple $(N, \nabla, h)$ is called a statistical manifold if $\nabla h$ is symmetric. If the curvature tensor $R$ of $\nabla$ vanishes, $(N, \nabla, h)$ is said to be flat.

For a statistical manifold $(N, \nabla, h)$, let $\nabla'$ be an affine connection on $N$ such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z) \quad \text{for } X, Y, Z \in \mathcal{X}(N),$$

where $\mathcal{X}(N)$ is the set of all tangent vector fields on $N$. The affine connection $\nabla'$ is torsion free, and $\nabla' h$ symmetric. Then $\nabla'$ is called the dual connection of $\nabla$, the triple $(N, \nabla', h)$ the dual statistical manifold of $(N, \nabla, h)$, and $(\nabla, \nabla', h)$ the dualistic structure on $N$. The curvature tensor of $\nabla'$ vanishes if and only if that of $\nabla$ does, and then $(\nabla, \nabla', h)$ is called the dually flat structure.
Let $N$ be a manifold with a dualistic structure $(\nabla, \nabla', h)$. For a real number $\alpha$, an affine connection defined by
\[
\nabla(\alpha) := \frac{1 + \alpha}{2} \nabla + \frac{1 - \alpha}{2} \nabla'
\]
is called an $\alpha$-connection of $(N, \nabla, h)$. The triple $(N, \nabla(\alpha), h)$ is also a statistical manifold, and $\nabla(-\alpha)$ the dual connection of $\nabla(\alpha)$. The 1-connection, the $(-1)$-connection and the 0-connection coincide with $\nabla$, $\nabla'$ and Levi-Civita connection of $(N, h)$, respectively. An $\alpha$-connection is not always flat [1].

If $(N, \nabla, h)$ is a flat statistical manifold, we call $\nabla(\alpha)$ an $\alpha$-transitively flat connection of $(N, \nabla, h)$. An $\alpha$-transitively flat connection is not always flat.

For $\alpha \in \mathbb{R}$, statistical manifolds $(N, \nabla, h)$ and $(\bar{N}, \bar{\nabla}, \bar{h})$ are said to be $\alpha$-conformally equivalent if there exists a function $\phi$ on $N$ such that
\[
\bar{h}(X, Y) = e^{\phi} h(X, Y),
\]
\[
h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \frac{1 + \alpha}{2} d\phi(Z) h(X, Y)
+ \frac{1 - \alpha}{2} \{ d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z) \}
\]
for $X, Y, Z \in \mathcal{X}(N)$. Two statistical manifolds $(N, \nabla, h)$ and $(\bar{N}, \bar{\nabla}, \bar{h})$ are $\alpha$-conformally equivalent if and only if the dual statistical manifolds $(N, \nabla', h)$ and $(\bar{N}, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. A statistical manifold $(N, \nabla, h)$ is called $\alpha$-conformally flat if $(N, \nabla, h)$ is locally $\alpha$-conformally equivalent to a flat statistical manifold [2].

3. $\alpha$-transitively flat connections and $\alpha$-conformal equivalence

We relate an $\alpha$-transitively flat connection of a flat statistical manifold with an $\alpha$-conformal equivalence of its statistical submanifold. Statistical submanifolds are defined in [4] and [6].

**LEMMA 3.1.** Let $(N, \nabla, h)$ be a flat statistical manifold, and $(M, D, g)$ a 1-conformally flat statistical submanifold realized in $(N, \nabla, h)$. Let $M_0$ be a simply connected open set of $M$. If $(M, D, g)$ is 1-conformally equivalent to a flat statistical manifold $(M_0, \bar{D}, \bar{g})$, $(M_0, D(\alpha), g)$ is $\alpha$-conformally equivalent to $(M_0, \bar{D}(\alpha), \bar{g})$, where $D(\alpha)$ the induced connection on $M_0$ by an $\alpha$-transitively flat connection $\nabla(\alpha)$ of $(N, \nabla, h)$, and $\bar{D}(\alpha)$ an $\alpha$-transitively flat connection of $(M_0, \bar{D}, \bar{g})$.

**Proof.** Let $D'$ and $\bar{D}'$ be the dual connection of $D$ and $\bar{D}$, respectively. Since $D(\alpha)$ is induced by $\nabla(\alpha)$,
\[
D(\alpha) = \frac{1 + \alpha}{2} D + \frac{1 - \alpha}{2} D' \quad \text{on } M_0
\]