Return to Equilibrium for Pauli-Fierz Systems

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Abstract. We study ergodic properties of Pauli-Fierz systems – $W^*$-dynamical systems often used to describe the interaction of a small quantum system with a bosonic free field at temperature $T \geq 0$. We prove that, for a small coupling constant uniform as the positive temperature $T \downarrow 0$, a large class of Pauli-Fierz systems has the property of return to equilibrium. Most of our arguments are general and deal with mathematical theory of Pauli-Fierz systems for an arbitrary density of bosonic field.

1 Introduction

A quantum system is often described by a $W^*$-algebra $\mathcal{M}$ with a $\sigma$-weakly continuous group of automorphisms $t \mapsto \tau^t$. The pair $(\mathcal{M}, \tau)$ is called a $W^*$-dynamical system and $\tau$ a $W^*$-dynamics. We say that the system $(\mathcal{M}, \tau)$ has the property of return to equilibrium if there exists a normal state $\omega$ on $\mathcal{M}$ such that for all normal states $\phi$ and $A \in \mathcal{M},$

$$\lim_{|t| \to \infty} \phi(\tau^t(A)) = \omega(A).$$

Such $\omega$ is obviously unique and $\tau$-invariant. Physical intuition suggests the following quasitheorem.

Quasitheorem Suppose that $(\mathcal{M}, \tau)$ describes a quantum system that is:

1. infinitely extended;
2. a localized perturbation of a thermal equilibrium system;
3. sufficiently regular;
4. sufficiently generic.

Then $(\mathcal{M}, \tau)$ has the property of return to equilibrium.

Conditions (1) and (3) are idealizations necessary to prove sharp mathematical results. In particular, it is well known that finite volume (confined) quantum systems do not return to equilibrium.

Condition (2) is related to the issue of stability of equilibrium states (see [BR2] and references therein). It is expected on physical grounds, and in some circumstances it can be proven, that if $(\mathcal{M}, \tau)$ describes a localized perturbation of a physical system away from thermal equilibrium, then there are no normal $\tau$-invariant states (see Subsections 3.6 and 7.9).
Concerning (4), some assumptions are necessary to prevent the existence of internal symmetries which would lead to an artificial multiplicity of \( \tau \)-invariant normal states. In our paper the conditions of this type will be called effective coupling conditions and they will be generically satisfied.

In this paper we will study a class of quantum systems which are commonly used to describe the interaction of a “small” quantum system, often called an “atom”, with a “bosonic reservoir”. We will call them Pauli-Fierz systems [PF]. They arise in physics as simplified versions of the non-relativistic QED.

We note that in the literature the name “Pauli-Fierz Hamiltonians” appears in a number of different (although closely related) contexts. Our definition of Pauli-Fierz systems is consistent with our previous work [DG, DJ1].

Our main result is a precise formulation of the conditions described in the “quasitheorem” and a proof that under these conditions Pauli-Fierz systems have the property of return to equilibrium. Results closely related to ours can be found in [BFS2, JP2, M] and we will discuss them in Subsection 1.2. The rest of this section is devoted to an informal discussion of our main results.

In our paper the small system is described by a finite-dimensional Hilbert space \( \mathcal{K} \) and a Hamiltonian \( K \).

The bosonic reservoir is described by a pair \((\mathcal{Z}, h)\) where \( \mathcal{Z} \) and \( h \) are the Hilbert space and the energy operator of a single boson. We will always assume that \( h \geq 0 \). Physically, the bosons can be interpreted as phonons or photons.

The interaction between the small system and the reservoir is specified by a form-factor \( \lambda v \), where \( v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z}) \) and \( \lambda \) is a real coupling constant which controls the strength of the interaction. Our main results hold for sufficiently small nonzero values of \( \lambda \).

The data \((\mathcal{K}, K, \mathcal{Z}, h, v)\) determine the basic Pauli-Fierz Hamiltonian, which is defined as the self-adjoint operator

\[
H = K \otimes 1 + 1 \otimes d\Gamma(h) + \lambda V
\]

on the Hilbert space \( \mathcal{H} = \mathcal{K} \otimes \Gamma_s(\mathcal{Z}) \), where \( \Gamma_s(\mathcal{Z}) \) is the bosonic Fock space over the 1-particle space \( \mathcal{Z} \) and the interaction term \( V \) is the field operator associated to the form-factor \( v \). Thus we obtain the \( W^* \)-dynamical system

\[
(\mathcal{B}(\mathcal{H}), e^{itH}, e^{-itH}).
\] (1.1)

The \( W^* \)-dynamical system (1.1) is however not our main object of study. We are interested in a family of \( W^* \)-dynamical systems that arise as thermodynamical limits of (1.1) and which describe Pauli-Fierz systems with non-zero radiation field density. Apart from \((\mathcal{K}, K, \mathcal{Z}, h, v)\), these systems are parameterized by a positive operator (the radiation density operator) \( \rho \) on \( \mathcal{Z} \) commuting with \( h \). We call them Pauli-Fierz systems at density \( \rho \). To describe such systems one needs to use the so-called Araki-Woods representations of CCR [AW, BR2]. In typical cases, for instance if \( \rho \) has some continuous spectrum, the corresponding \( W^* \)-algebras are of