Exact Solution of the AVZ-Hamiltonian in the Grand-Canonical Ensemble

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Abstract. The thermodynamic behavior of the Angelescu-Verbeure-Zagrebnov (AVZ) Hamiltonian [1], also called the superstable Bogoliubov model, is solved for any temperature and any chemical potential. It is found that its thermodynamics coincides with one for the Mean-Field Gas for small chemical potential or high temperature. However, for large chemical potential or low temperature, a non-conventional Bose condensation appears with, even at zero-temperature, a (non-zero) particle density outside the condensate. Following [2], the analysis in the present paper corresponds to the main technical step to deduce, in the canonical ensemble, a new microscopic theory of superfluidity at all temperatures explained in [3].

1 Introduction

Let an interacting homogeneous gas of $n$ spinless bosons with mass $m$ be enclosed in a cubic box $\Lambda = \frac{3}{\alpha_1} L \subset \mathbb{R}^3$. We denote by $\varphi (x) = \varphi (\| x \|)$ a (real) two-body interaction potential and we assume that:

(A) $\varphi (x) \in L^1 (\mathbb{R}^3)$.

(B) Its (real) Fourier transformation

$$\lambda_k = \int_{\mathbb{R}^3} d^3 x \varphi (x) e^{-ikx}, \ k \in \mathbb{R}^3,$$

satisfies: $\lambda_0 > 0$ and $0 \leq \lambda_k = \lambda_{-k} \leq \lim_{\| k \| \to 0^+} \lambda_k$ for $k \in \mathbb{R}^3$.

(C) The interaction potential $\varphi (x)$ satisfies:

(C1) : $\frac{\lambda_0}{2} + g_{00} \geq 0$, or (C2) : $\frac{\lambda_0}{2} + g_{00} < 0$,

where the (effective coupling) constant $g_{00}$ equals

$$g_{00} \equiv -\frac{1}{4 (2\pi)^3} \int_{\mathbb{R}^3} d^3 k \frac{\lambda_k^2}{\varepsilon_k} < 0,$$  \hspace{1cm} (1.1)

with the one-particle energy spectrum defined by $\varepsilon_k \equiv \hbar^2 k^2 / 2m$.

The last conditions (C1)–(C2) will be important at the end of this paper and first appeared in the study of the Weakly Imperfect Bose Gas [1,4–6].
The (non-diagonal) AVZ-Hamiltonian [1], also called the superstable Bogoliubov Hamiltonian, is defined for \( \lambda_0 > 0 \) by
\[
H_{A, \lambda_0}^{SB} = H_{A, 0}^B + U_{A}^{MF},
\]
where
\[
U_{A}^{MF} \equiv \frac{\lambda_0}{2V} \sum_{k_1, k_2 \in \Lambda^*} a_{k_1}^* a_{k_2} a_{k_2} a_{k_1} = \frac{\lambda_0}{2V} \left( N_\Lambda^2 - N_\Lambda \right),
\]
\[
H_{A, \lambda_0}^B \equiv T_{\Lambda} + U_{\Lambda}^D + U_{\Lambda}^{ND} + U_{\Lambda}^{BMF},
\]
and
\[
N_\Lambda \equiv \sum_{\mathbf{k} \in \Lambda^*} a_{\mathbf{k}}^* a_{\mathbf{k}}, \quad T_{\Lambda} \equiv \sum_{\mathbf{k} \in \Lambda^*} \varepsilon_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}},
\]
\[
U_{\Lambda}^D \equiv \frac{1}{2V} \sum_{\mathbf{k} \in \Lambda^* \setminus \{0\}} \lambda_\mathbf{k} a_{\mathbf{0}}^* a_{\mathbf{k}}^* \left( a_{\mathbf{k}}^* a_{\mathbf{0}} + a_{-\mathbf{k}}^* a_{-\mathbf{0}} \right),
\]
\[
U_{\Lambda}^{ND} \equiv \frac{1}{2V} \sum_{\mathbf{k} \in \Lambda^* \setminus \{0\}} \lambda_\mathbf{k} \left( a_{\mathbf{k}}^* a_{-\mathbf{0}}^* a_{\mathbf{0}}^2 + a_{\mathbf{0}}^* a_{\mathbf{k}}^* a_{-\mathbf{k}}^* \right),
\]
\[
U_{\Lambda}^{BMF} \equiv \frac{\lambda_0}{2V} a_{\mathbf{0}}^* a_{\mathbf{0}}^2 + \frac{\lambda_0}{V} a_{\mathbf{0}}^* a_{\mathbf{0}} \sum_{\mathbf{k} \in \Lambda^* \setminus \{0\}} a_{\mathbf{k}}^* a_{\mathbf{k}}.
\]
The Hamiltonian \( H_{A, \lambda_0}^{SB} \) acts on the boson Fock space
\[
\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)},
\]
with \( \mathcal{H}_B^{(n)} \) defined as the symmetrized \( n \)-particle Hilbert spaces
\[
\mathcal{H}_B^{(n)} \equiv \left( L^2 (\Lambda^*) \right)_{\text{symm}}, \quad \mathcal{H}_B^{(0)} \equiv \mathbb{C},
\]
see [7,8]. Using periodic boundary conditions, let
\[
\Lambda^* \equiv \left\{ \mathbf{k} \in \mathbb{R}^3 : \ k_\alpha = \frac{2\pi n_\alpha}{L}, \ n_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, 2, 3 \right\}
\]
be the set of wave vectors. Also, note that \( a_{\mathbf{k}}^\# = \{ a_{\mathbf{k}}^* \text{ or } a_{\mathbf{k}} \} \) are the usual boson creation / annihilation operators in the one-particle state \( \psi_{\mathbf{k}}(x) = V^{-\frac{1}{2}} e^{i k_x x}, k \in \Lambda^*, x \in \Lambda, \) acting on the boson Fock space \( \mathcal{F}_\Lambda^B. \) Under assumptions (A) and (B) on the interaction potential \( \varphi(x) \) the Hamiltonian \( H_{A, \lambda_0}^{SB} \) is superstable [8].

To fix the notations, \( \beta > 0 \) is the inverse temperature, \( \mu \) the chemical potential, \( \rho > 0 \) the fixed full particle density.

Before we embark on the rigorous results of this model, it may be useful to give briefly its origin and history.