Semiclassical $L^p$ Estimates

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Abstract. The purpose of this paper is to use semiclassical analysis to unify and generalize $L^p$ estimates on high energy eigenfunctions and spectral clusters. In our approach these estimates do not depend on ellipticity and order, and apply to operators which are selfadjoint only at the principal level. They are estimates on weakly approximate solutions to semiclassical pseudodifferential equations.

1. Introduction

To motivate our results let us first recall Sogge’s $L^p$ estimate [18] on spectral clusters, $\Pi_{[\lambda,\lambda+1]}$, of the Laplace–Beltrami operator, $-\Delta_g$, on a compact Riemannian manifold, $(M^n, g)$:

$$\Pi_{[\lambda,\lambda+1]} = O(\lambda^2) : L^2(M, d\text{vol}_g) \longrightarrow L^p(M, d\text{vol}_g), \quad p = \frac{2(n+1)}{n-1}, \quad (1.1)$$

where

$$\Pi_I \overset{\text{def}}{=} \sum_{\lambda_j \in I} u_j \otimes \bar{u}_j, \quad -\Delta_g u_j = \lambda_j^2 u_j, \quad \|u_j\|_{L^2(M, d\text{vol}_g)} = 1,$$

and $\{u_j\}$ form a complete orthonormal set.

The spectral counting remainder estimates of Avakumović–Levitan–Hörmander implies a bound $\Pi_{[\lambda,\lambda+1]} = O(\lambda^{(n-1)/2}) : L^2 \rightarrow L^\infty$. Combining this with (1.1) and the trivial estimate, $\Pi_{[\lambda,\lambda+1]} = O(1) : L^2 \rightarrow L^2$, we obtain optimal $L^2 \rightarrow L^p$ bounds for the spectral cluster operator (see the continuous line in Figure 1).

A similar problem was considered for the harmonic oscillator, $-\Delta + |x|^2$ in $\mathbb{R}^n$, by Karadzhov, Thangavelu, and the first two authors – see [11] and references given there. In that case, and for $n \geq 2$,

$$\Pi_{[\lambda,\lambda+1]} = \begin{cases} O(1) & : L^2(\mathbb{R}^n) \longrightarrow L^{2n/(n-2)}(\mathbb{R}^n), \\ O((\lambda-1 \log(n+1)/\lambda)^1/(n+3)) & : L^2(\mathbb{R}^n) \longrightarrow L^{2(n+3)/(n+1)}(\mathbb{R}^n), \end{cases} \quad (1.2)$$
where now
\[ \Pi_I \overset{\text{def}}{=} \sum_{\lambda_j \in I} u_j \otimes \bar{u}_j, \quad (-\Delta + |x|^2)u_j = \lambda_j^2 u_j, \quad \|u_j\|_{L^2(\mathbb{R}^n)} = 1, \]
and again \( \{u_j\} \) form a complete orthonormal set. An interpolated result without the logarithmic growth is also valid (see the dashed and dotted lines in Figure 1). Strichartz estimates [9,19] lie at the heart of estimates (1.1) and (1.2). In fact, a quick proof of the first estimate in (1.2) follows from the pointwise decay of the Schrödinger propagator and the end-point Strichartz estimate of Keel and Tao [9].

A semiclassical point of view – see [3, 4, 12] – allows to put both results in the same setting. For compact manifolds we consider the family of operators \(-\hbar^2 \Delta_g - 1, \ h \sim \lambda^{-1}\), and for the harmonic oscillator, \(-\hbar^2 \Delta_y + |y|^2 - 1\), where now \( h \sim \lambda^{-2} \), and \( y = \hbar^{1/2} x \) (see Example 1 below).

A natural generalization of the problem can then be formulated as follows: suppose that \( P \) is a semiclassical quantization of a classical observable \( p \), that is \( P \) is a semiclassical pseudodifferential operator with the principal symbol given by \( p \). Under what conditions on \( p \) and for what \( \mu(q) \) do we have
\[ Pu = \mathcal{O}_{L^2}(h), \quad \|u\|_2 = 1 \implies \|u\|_q = \mathcal{O}(h^{-\mu(q)}) \quad (1.3) \]

Here the family of functions \( u = u(h) \) is assumed to be localized in phase space:
\[ \exists K \subset \mathbb{R}^n, \quad \chi \in C_c^\infty(\mathbb{R}^n), \quad \text{independent of } h, \text{ such that} \]
\[ \text{supp } u(h) \subset K \quad \text{and} \quad \forall k, \quad u(h) = \chi(hD)u(h) + \mathcal{O}(h^k). \quad (1.4) \]

An important comment is that the approximate solutions (1.3) are local, that is, the statement \( Pu = \mathcal{O}_{L^2}(h) \), is invariant under localization in position \((x)\) and in momentum \((hD)\).

In this introduction we state our results for the generalized Schrödinger operator,
\[ P = -\hbar^2 \Delta_g + V(x), \quad V \in C^\infty(\mathbb{R}^n; \mathbb{R}), \quad \Delta_g = \frac{1}{g} \sum_{i,j=1}^{n} \partial_{x_i} \sqrt{g} g^{ij} \partial_{x_j}, \quad (1.5) \]
where \( g \overset{\text{def}}{=} (g^{ij}(x))_{i,j=1}^{n} \) is a non-degenerate matrix, and \( \bar{g} \overset{\text{def}}{=} |\det g^{-1}| \). The more general results will be presented in the sections below. The proofs are based on semiclassical developments of the ideas from [10, 11]. However, except for the use of basic aspects of semiclassical analysis reviewed in Section 2 and one application of the end point Strichartz estimate of Keel and Tao [9] the paper is self contained.

**Theorem 1.** Suppose that \( P \) is given by (1.5), \( u = u(h) \) satisfies (1.4), and
\[ Pu = \mathcal{O}_{L^2}(h), \quad \|u\|_{L^2} = 1. \quad (1.6) \]
Then
\[ \|u\|_p = \mathcal{O}(h^{-\frac{2}{n}}), \quad p = \frac{2n}{n-2}, \quad n > 2, \quad (1.7) \]