Semiclassical Analysis for Magnetic Scattering by Two Solenoidal Fields: Total Cross Sections

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Abstract. The fact that vector potentials have a direct significance to quantum particles moving in magnetic fields is known as the Aharonov–Bohm effect (A–B effect). We study this quantum effect through the semiclassical analysis on total cross sections in the magnetic scattering by two solenoidal (point-like) fields with total flux vanishing in two dimensions. We derive the asymptotic formula with first three terms. The system with two parallel fields seems to be important in practical aspects as well as in theoretical aspects, because it may be thought of a toroidal solenoid with zero cross section in three dimensions under the idealization that the two fields connect at infinity in their direction. The corresponding classical mechanical system has the trajectory oscillating between two centers of fields. The special emphasis is placed on analyzing how the trapping effect from classical mechanics is related to the A–B quantum effect in the semiclassical asymptotic formula.

1. Introduction

In quantum mechanics, vector potentials have a direct significance to particles moving in magnetic fields. This is known as the Aharonov–Bohm effect (A–B effect) [3]. The aim of the present work is to study this quantum effect through the semiclassical analysis on total cross sections of magnetic scattering by two solenoidal (point-like) fields with total flux vanishing in two dimensions. Such a system seems to be important in practical aspects as well as in theoretical aspects, because it may be thought of a toroidal solenoid with zero cross section in three dimensions under the idealization that two parallel fields connect at infinity in their direction (direction perpendicular to the plane).

Let $\Lambda(x) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector potential defined by

$$\Lambda(x) = \left( -x_2/|x|^2, x_1/|x|^2 \right) = \left( -\partial_2 \log |x|, \partial_1 \log |x| \right), \quad \partial_j = \partial/\partial x_j.$$
The potential has the point-like field
\[ \nabla \times \Lambda = (\partial^2_1 + \partial^2_2) \log |x| = \Delta \log |x| = 2\pi \delta(x) \]
at the origin. The particle moving in two solenoidal fields with flux \( \pm \alpha \in \mathbb{R} \) at center \( e_{\pm} \in \mathbb{R}^2 \) is governed by the Hamiltonian
\[ H_h = (-i\hbar \nabla - A)^2 = \sum_{j=1,2} (-i\hbar \partial_j - a_j)^2, \quad 0 < h \ll 1, \quad (1.1) \]
where \( A(x) = (a_1(x), a_2(x)) : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) takes the form
\[ A(x) = \alpha \Lambda(x - e_+) - \alpha \Lambda(x - e_-), \quad e_+ \neq e_- . \]
The potential \( A(x) \) has a strong singularity at centers \( e_{\pm} \) of two fields. For this reason, \( H_h \) formally defined by (1.1) is not necessarily essentially self-adjoint in \( C_0^\infty(\mathbb{R}^2 \setminus \{e_-, e_+\}) \). We have to impose boundary conditions at \( e_{\pm} \) to define a self-adjoint realization in the space \( L^2 = L^2(\mathbb{R}^2) \). We know that \( H_h \) admits the self-adjoint realization under the boundary conditions
\[ \lim_{|x| \rightarrow 0} |u(x)| < \infty . \quad (1.2) \]
We denote by the same notation \( H_h \) this self-adjoint operator and by \( f_h(\omega_+ \rightarrow \omega) \) the amplitude for the scattering from incident direction \( \omega_+ \in S^1 \) to final one \( \omega \) at energy \( E > 0 \), where \( S^1 \) denotes the unit circle and \( E \) is fixed throughout. When the total flux vanishes, \( A(x) \) falls off like \( A(x) = O(|x|^{-2}) \) at infinity, so that the forward amplitude \( f_h(\omega_+ \rightarrow \omega) \) is finite and the total scattering cross section
\[ \sigma_h(\omega_+) = \int |f_h(\omega_+ \rightarrow \omega)|^2 d\omega \]
is also well defined, where the integration with no domain attached is taken over the whole space. We often use this abbreviation. In the present work we derive the asymptotic formula with first three terms for \( \sigma_h(\omega_+) \) in the semiclassical limit \( h \rightarrow 0 \). The system with two solenoidal fields has the trajectory oscillating between two centers \( e_{\pm} \). Our motivation comes from analyzing how this trapping effect is reflected in the asymptotic formula through the A–B effect.

The asymptotic formula is described in terms of amplitudes for the scattering by a single solenoidal field. We begin by making a quick review on it. The system with single field is exactly solvable, and the explicit representation for amplitude has been calculated by [3] (see [1, 2, 5, 15] also). We now consider the Hamiltonian
\[ H_{\alpha h} = (-i\hbar \nabla - \alpha \Lambda)^2, \quad \Lambda = \Lambda(x), \quad (1.3) \]
which governs the particle moving in the point-like field \( 2\alpha \delta(x) \) at the origin. For the same reason as above, this operator is not essentially self-adjoint in \( C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \). In fact, it is known to be a symmetric operator with type \( (2, 2) \) of deficiency indices \([1, 5]\). A self-adjoint extension \( H_{\alpha h} \) is obtained by imposing the boundary condition
\[ \lim_{|x| \rightarrow 0} |u(x)| < \infty . \quad (1.4) \]