Bifurcations of Positive and Negative Continua in Quasilinear Elliptic Eigenvalue Problems

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Abstract. The main result of this work is a Dancer-type bifurcation result for the quasilinear elliptic problem

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u + h(x, u(x); \lambda) \quad \text{in } \Omega; \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(P)

Here, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) (\(N \geq 1\)), \(\Delta_p u \stackrel{\text{def}}{=} \text{div}(\nabla u |^{p-2} \nabla u)\) denotes the Dirichlet \(p\)-Laplacian on \(W^{1,p}_0(\Omega)\), \(1 < p < \infty\), and \(\lambda \in \mathbb{R}\) is a spectral parameter. Let \(\mu_1\) denote the first (smallest) eigenvalue of \(-\Delta_p\). Under some natural hypotheses on the perturbation function \(h: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), we show that the trivial solution \((0, \mu_1) \in E = W^{1,p}_0(\Omega) \times \mathbb{R}\) is a bifurcation point for problem (P) and, moreover, there are two distinct continua, \(Z^{\mu_1}_+\) and \(Z^{\mu_1}_-\), consisting of nontrivial solutions \((u, \lambda) \in E\) to problem (P) which bifurcate from the set of trivial solutions at the bifurcation point \((0, \mu_1)\). The continua \(Z^{\mu_1}_+\) and \(Z^{\mu_1}_-\) are either both unbounded in \(E\), or else their intersection \(Z^{\mu_1}_+ \cap Z^{\mu_1}_-\) contains also a point other than \((0, \mu_1)\). For the semilinear problem (P) (i.e., for \(p = 2\)) this is a classical result due to E. N. Dancer from 1974. We also provide an example of how the union \(Z^{\mu_1}_+ \cap Z^{\mu_1}_-\) looks like (for \(p > 2\)) in an interesting particular case.

Our proofs are based on very precise, local asymptotic analysis for \(\lambda\) near \(\mu_1\) (for any \(1 < p < \infty\)) which is combined with standard topological degree arguments from global bifurcation theory used in Dancer’s original work.

1. Introduction

This work is concerned with bifurcations of continua of “positive” and “negative” solutions to quasilinear elliptic problems of the following type:

\[
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u + h(x, u(x); \lambda) \quad \text{in } \Omega; \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.1)
Here, $\Omega$ denotes a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $\Delta_p$ stands for the Dirichlet $p$-Laplacian defined by $\Delta_p u \overset{\text{def}}{=} \text{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$, $\lambda$ ($\lambda \in \mathbb{R}$) serves as a bifurcation parameter, and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $h(x, \cdot, \cdot)$ continuous for a.e. $x \in \Omega$. When considering bifurcations from a trivial solution, we naturally assume also $h(x, 0; \lambda) = 0$ and $h(x, u; \lambda)/|u|^{p-1} \to 0$ as $u \to 0$, pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$. A trivial solution of (1.1) is any pair $(0, \lambda) \in E \overset{\text{def}}{=} W^{1,p}_0(\Omega) \times \mathbb{R}$.

In analogy with classical results of Dancer [10] for the semilinear case $p = 2$, our main goal is to show the existence of two distinct continua of nontrivial (weak) solutions to problem (1.1), “positive” and “negative” ones, that bifurcate from the set of trivial solutions at the point $(0, \mu_1)$ in the positive and negative directions $\varphi_1$ and $-\varphi_1$, respectively (see Lemma 3.6). As usual, $\mu_1$ denotes the first (smallest) eigenvalue of $-\Delta_p$ which is known to be simple with a positive eigenfunction $\varphi_1 \in W^{1,p}_0(\Omega)$. Under a continuum in a Banach space we mean a closed connected set which contains at least two distinct points. Similarly to bifurcations from zero we treat also bifurcations from infinity under the condition $h(x, u; \lambda)/|u|^{p-1} \to 0$ as $|u| \to \infty$, pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

To be more specific about our present results, let us begin by considering the semilinear case $p = 2$ first: The classical global bifurcation result of Rabinowitz [31, Theorem 1.3] exhibits a continuum of nontrivial solutions to problem (1.1) which emanates from the set of trivial solutions at the bifurcation point $(0, \mu_1)$. Furthermore, Dancer’s result [10, Theorem 2] guarantees the bifurcation of two continua of “positive” and “negative” solutions to problem (1.1) in the directions $\pm \varphi_1$. Indeed, in a sufficiently small neighborhood of $(0, \mu_1)$ these continua contain only solutions $(u, \lambda) \in E$ of problem (1.1) satisfying $u = \tau(\varphi_1 + v^\top)$ where $\tau \in \mathbb{R}$ and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $\tau \to 0$. Hence, $u > 0$ in $\Omega$ ($u < 0$ in $\Omega$, respectively) if and only if $\tau > 0$ ($\tau < 0$), provided $|\tau| > 0$ is small enough.

Now let us consider the quasilinear case $p \neq 2$. The analogue of Rabinowitz’ result [31, Theorem 1.3] for problem (1.1) has been obtained in del Pino and Manásevich [30] with a continuum of nontrivial solutions bifurcating from the point $(0, \mu_1)$ and having the same properties as in the case $p = 2$. In the work reported here we obtain the corresponding analogue (Theorem 3.7 below) of Dancer’s result [10, Theorem 2] for $1 < p < \infty$. We treat problems with a more general $(p-1)$-homogeneous part than just (1.1) treated in [30]. Similarly to a bifurcation from zero at $(0, \mu_1)$ sketched above, under a bifurcation from infinity at $(+\infty, \mu_1)$, respectively) we mean a continuum of solutions $(u, \lambda) \in E$ of problem (1.1) satisfying $u = t^{-1}(\varphi_1 + v^\top)$ where $0 \neq t \in \mathbb{R}$ and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $t \to 0$. Again, $u > 0$ in $\Omega$ ($u < 0$ in $\Omega$, respectively) if and only if $t > 0$ ($t < 0$), provided $|t| > 0$ is small enough.

In an analogy with the case $p = 2$, we use the fact that $\mu_1$ is a simple eigenvalue of $-\Delta_p$ with a positive eigenfunction $\varphi_1$ in an essential way. Under a rather restrictive hypothesis, this extension of Dancer’s result has already been stated in Drábek [14, Theorem 14.20, p. 191] without proof. His hypothesis [14,