Constructive $\phi^4$ Field Theory without Tears

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Abstract. We propose to treat the $\phi^4$ Euclidean theory constructively in a simpler way. Our method, based on a new kind of “loop vertex expansion”, no longer requires the painful intermediate tool of cluster and Mayer expansions.

1. Introduction

Constructive field theory builds functions whose Taylor expansion is perturbative field theory [15, 24]. Any formal power series being asymptotic to infinitely many smooth functions, perturbative field theory alone does not give any well defined mathematical recipe to compute to arbitrary accuracy any physical number, so in a deep sense it is no theory at all.

In field theory “thermodynamic” or infinite volume quantities are expressed by connected functions. One main advantage of perturbative field theory is that connected functions are simply the sum of the connected Feynman graphs. But the expansion diverges because there are too many such graphs. However to know connectedness does not require the full knowledge of a Feynman graph (with all its loop structure) but only the (classical) notion of a spanning tree in it. This remark is at the core of the developments of constructive field theory, such as cluster expansions, summarized in the constructive golden rule:

“Thou shall not know most of the loops, or thou shall diverge!”

Some time ago Fermionic constructive theory was quite radically simplified. It was realized that it is possible to rearrange perturbation theory order by order by grouping together pieces of Feynman graphs which share a common tree [1, 22]. This is made easily with the help of a universal combinatoric so-called forest formula [2, 5] which once and for all essentially solves the problem that a graph can have many spanning trees. Indeed it splits any amplitude of any connected graph in a certain number of pieces and attributes them in a “democratic” and “positivity preserving” way between all its spanning trees. Of course the possibility for such a rearrangement to lead to convergent resummation of Fermionic perturbation
theory ultimately stems from the Pauli principle which is responsible for analyticity of that expansion in the coupling constant.

Using this formalism Fermionic theory can now be manipulated at the constructive level almost as easily as at the “perturbative level to all orders”. It leads to powerful mathematical physics theorems such as for instance those about the behavior of interacting Fermions in 2 dimensions [8,11,25], and to more explicit constructions [9] of just renormalizable Fermionic field theories such as the Gross–Neveu model in two dimensions first built in [13,14].

But Bosonic constructive theory remained awfully difficult. To compute the thermodynamic functions, until today one needed to introduce two different expansions one of top of the other. The first one, based on a discretization of space into a lattice of cubes which breaks the natural rotation invariance of the theory, is called a cluster expansion. The result is a dilute lattice gas of clusters but with a remaining hardcore interaction. Then a second expansion called Mayer expansion removes the hardcore interaction. The same tree formula is used twice once for the cluster and once for the Mayer expansion\(^1\), the breaking of rotation invariance to compute rotation invariant quantities seems ad hoc and the generalization of this technique to many renormalization group steps is considered so difficult that despite courageous attempts towards a better, more explicit formalization [4,6], it remains until now confined to a small circle of experts.

The Bosonic constructive theory cannot be simply rearranged in a convergent series order by order as in the Fermionic case, because all graphs at a given order have the same sign. Perturbation theory has zero convergence radius for bosons. The oscillation which allows resummation (but only, e.g., in the Borel sense) of the perturbation theory must take place between infinite families of graphs of different orders. To explicitly identify such families and rearrange the perturbation theory accordingly seemed until now very difficult. The cluster and Mayer expansion perform this task but in a very complicated and indirect way.

In this paper we at last identify such infinite families of graphs. They give rise to an explicit convergent expansion for the connected functions of Bosonic \(\phi^4\) theory, without any lattice and cluster or Mayer expansion. In fact we stumbled upon this new method by trying to adapt former cluster expansions to large matrix \(\phi^4\) models in order to extend constructive methods to non-commutative field theory (see [26] for a recent review). The matrix version is described in a separate publication [27]. Hopefully it should allow a non-perturbative construction of the \(\phi^4\) theory on Moyal space \(\mathbb{R}^4\), whose renormalizable version was pioneered by Grosse and Wulkenhaar [16].

2. The example of the pressure of \(\phi^4\)

We take as first example the construction of the pressure of \(\phi^4\) in a renormalization group (RG) slice. The goal is, e.g., to prove its Borel summability in the coupling

\(^1\)It is possible to combine both expansions into a single one [3], but the result cannot be considered a true simplification.