Decorrelation Estimates for a 1D Tight Binding Model in the Localized Regime

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Abstract. In this article, we prove decorrelation estimates for the eigenvalues of a 1D discrete tight-binding model near two distinct energies in the localized regime. Consequently, for any integer $n \geq 2$, the asymptotic independence for local level statistics near $n$ distinct energies is obtained.

1. Introduction

The present paper deals with the following lattice Hamiltonian with off-diagonal disorder in dimension 1: for $u = \{u(n)\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$, set

$$H_\omega u(n) = \omega_n(u(n) - u(n + 1)) - \omega_{n-1}(u(n-1) - u(n)). \quad (1.1)$$

The model (1.1) appears in the description of waves (light, acoustic waves, etc) which propagate through a disordered, discrete medium (c.f. [3] and references therein). We can see $\{\omega_n\}_{n \in \mathbb{Z}}$ in this model as weights of bonds of the lattice $\mathbb{Z}$.

Throughout this article, we assume that $\omega := \{\omega_n\}_{n \in \mathbb{Z}}$ are non-negative i.i.d. random variables (r.v.’s for short) with a bounded, compactly supported density $\rho$.

In addition, from Sects. 1–3, we assume more that $\omega_n \in [\alpha_0, \beta_0]$ for all $n \in \mathbb{Z}$ where $\beta_0 > \alpha_0 > 0$. In Sect. 4, we will comment on relaxing the hypothesis of the lower bound of r.v.’s $\omega$.

It is known that (see [3])

- the operator $H_\omega$ admits an almost sure spectrum $\Sigma := [0, 4\beta_0]$. 
- $H_\omega$ has an integrated density of states defined as follows: 
  $$\omega-a.s., \text{the following limit exists and is } \omega \text{ independent:}$$

$$N(E) := \lim_{|\Lambda| \to +\infty} \frac{\# \{ \text{e.v. of } H_\omega \text{ less than } E \} }{|\Lambda|} \quad \text{for a.e. } E. \quad (1.2)$$
As a direct consequence of the Wegner estimate (see Theorem 2.1 in Sect. 2), $N(E)$ is defined everywhere in $\mathbb{R}$ and absolutely continuous w.r.t. Lebesgue measure with a bounded derivative $\nu(E)$ called the density of states of $H_\omega$.

In the present paper, we follow a usual way to study various statistics related to random operators. We restrict the operator $H_\omega$ on some interval $\Lambda \subset \mathbb{Z}$ of finite length with some boundary condition and obtain a finite-volume operator which is denoted by $H_\omega(\Lambda)$. Then, we study diverse statistics for this operator in the limit when $|\Lambda|$ goes to infinity.

Throughout this paper, the boundary condition to define $H_\omega(\Lambda)$ is always the periodic boundary condition. For example, if $\Lambda = [1,N]$, the operator $H_\omega(\Lambda)$ is a symmetric $N \times N$ matrix of the following form:

$$
\begin{pmatrix}
\omega_N + \omega_1 & -\omega_1 & 0 & \ldots & 0 & -\omega_N \\
-\omega_1 & \omega_1 + \omega_2 & -\omega_2 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & \omega_{N-2} + \omega_{N-1} & -\omega_{N-1} \\
-\omega_N & 0 & 0 & \ldots & -\omega_{N-1} & \omega_{N-1} + \omega_N \\
\end{pmatrix}.
$$

For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L := [-L,L]$ be a large interval in $\mathbb{Z}$ and $|\Lambda| := (2L+1)$ be its cardinality.

We will denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega,\Lambda) \leq E_2(\omega,\Lambda) \leq \cdots \leq E_{|\Lambda|}(\omega,\Lambda)$.

Let $I$ be the localized regime (the region of localization) in $\Sigma$ where the finite-volume fractional-moment criteria for localization are satisfied for the finite-volume operators $H_\omega(\Lambda)$ when $|\Lambda|$ is large enough (see Section 2 and [1] for more details). In this region, the spectrum of $H_\omega$ is pure point and the corresponding eigenfunctions decay exponentially at infinity.

Pick $E$ a positive energy in $I$ with $\nu(E) > 0$ and define the local level statistics near $E$ as follows:

$$
\Xi(\xi, E, \omega, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi)
$$

where

$$
\xi_n(E, \omega, \Lambda) = |\Lambda|\nu(E)(E_n(\omega, \Lambda) - E).
$$

For the model (1.1), it is known that the weak limit of the above point process is a Poisson point process:

**Theorem 1.1.** [3] Assume that $E$ is a positive energy in $I$ with $\nu(E) > 0$.

Then, when $|\Lambda| \to +\infty$, the point process $\Xi(\xi, E, \omega, \Lambda)$ converges weakly to a Poisson point process with the intensity 1, i.e., for $(U_j)_{1 \leq j \leq J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_j \cap U_j' = \emptyset$ if $j \neq j'$ and $(k_j)_{1 \leq j \leq J} \in \mathbb{N}^J$, we have

$$
\lim_{|\Lambda| \to +\infty} \mathbb{P} \left( \left\{ \begin{array}{c}
\sum_{j=1}^J \mathbb{1}_{\{\xi_j(E, \omega, \Lambda) \in U_1\}} = k_1 \\
\vdots \\
\sum_{j=1}^J \mathbb{1}_{\{\xi_j(E, \omega, \Lambda) \in U_J\}} = k_J
\end{array} \right\} \right) - \prod_{j=1}^J \frac{|U_j| k_j}{k_j!} e^{-|U_j|} = 0.
$$