Resonant Delocalization on the Bethe Strip

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Abstract. Recently, Aizenman and Warzel discovered a mechanism for the appearance of absolutely continuous spectrum for random Schrödinger operators on the Bethe lattice through rare resonances (resonant delocalization). We extend their analysis to operators with matrix-valued random potentials drawn from ensembles such as the Gaussian Orthogonal Ensemble. These operators can be viewed as random operators on the Bethe strip, a graph (lattice) with loops.

1. Introduction

Let $\mathcal{T}$ be a regular rooted tree with branching number $K > 1$ (Bethe lattice). We shall be interested in random Schrödinger operators on the Cartesian product $\mathcal{T} \times G$ of $\mathcal{T}$ and a finite graph $G$ with $W$ vertices (Bethe strip). Equivalently, these can be seen as random Schrödinger operators on $\mathcal{T}$ with matrix-valued potential. The precise definition is as follows: $H = H_{\lambda, \omega}$ is a random operator acting on

$$\ell^2(\mathcal{T} \times G) = \ell^2(\mathcal{T} \to \mathbb{R}^W),$$

and given by the matrix elements

$$H_{\lambda, \omega}(x, y) = \begin{cases} 
1_{W \times W}, & x \sim y \ (x \text{ is adjacent to } y) \\
A + \lambda V_\omega(x), & x = y \\
0, & \text{otherwise}
\end{cases}, \quad x, y \in \mathcal{T}. \quad (1)$$

Here $\lambda \geq 0$ is a coupling constant, $\omega$ denotes an element of the probability space, $A$ is a fixed $W \times W$ Hermitian matrix, and $V_\omega(x)$ are independent identically distributed $W \times W$ random matrices. The potential $A + \lambda V_\omega(x)$ will be denoted $U_\omega(x)$.

The question that we shall address is, what is the spectral type of $H$ when $\lambda$ is small. Before stating our results, let us review what was previously known.

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For the Bethe lattice \((W = 1, A = 0)\) in our notation, the spectrum of the unperturbed operator \((\lambda = 0)\) is purely absolutely continuous and fills the interval \([-2\sqrt{K}, 2\sqrt{K}]\). Under mild assumptions on the potential, Klein [9–11] showed that, for small \(\lambda > 0\), the spectrum in \([-2\sqrt{K} + \epsilon, 2\sqrt{K} - \epsilon]\) is also (almost surely) absolutely continuous. Additional proofs and generalizations of this result were found by Aizenman et al. [3] and by Froese et al. [7].

On the other hand, Aizenman [1] proved that, for small \(\lambda\), the spectrum of \(H\) outside \([-K - 1 + \epsilon, K + 1 + \epsilon]\) is almost surely pure point.

In the recent work, Aizenman and Warzel [4] proved the presence of absolutely continuous spectrum throughout the interval \([-K - 1 + \epsilon, K + 1 - \epsilon]\). They found a new mechanism for the appearance of absolutely continuous spectrum, entirely different from the one appearing inside the spectrum of the unperturbed operator, and coined the term “resonant delocalization” for it. As opposed to the absolutely continuous spectrum in the interval \([-2\sqrt{K}, 2\sqrt{K}]\), which appears due to the stability of the absolutely continuous spectrum on the Bethe lattice, the absolutely continuous spectrum in \([-K - 1, K + 1]\) \([-2\sqrt{K}, 2\sqrt{K}]\) (in the Lifshitz tails) appears due to resonances between distant sites. The interval \([K - 1, K + 1]\) is exactly the \(\ell^1\) spectrum of the unperturbed operator; the importance of the \(\ell^1\) spectrum is further discussed in [4] and in the survey by Warzel [15].

The goal of this present work was to extend the result of [4] to the case \(W > 1\) of the Bethe strip. We make use of significant parts of the work [4]; for the reader’s convenience, we denote by Statement X* the generalization of [4, Statement X].

Denote by \(\{\nu_i\}_{i=1}^{W}\) the eigenvalues of \(A\), and let

\[
S_{\epsilon} = \bigcup_i \left[\nu_i - (K + 1) + \epsilon, \nu_i + (K + 1) - \epsilon\right].
\]

Our main result is

**Theorem 1.1** (Corollary 2.3*). Assume that \(V_\omega(x)\) are drawn from the Gaussian Orthogonal Ensemble (GOE). For any \(\epsilon > 0\) any open interval \(I \subset S_{\epsilon}\) almost surely has absolutely continuous spectrum of \(H_{\lambda, \omega}\) in it, when \(\lambda > 0\) is sufficiently small.

Thus the mechanism of resonant delocalization from [4] may be extended to the Bethe strip, a lattice with loops. See [15, Sect. 4] for a more general discussion of possible further extensions.

Theorem 1.1 should also be compared with the result of Klein and Sadel [12] (and its ramification [13]), who proved, under weaker assumptions on the potential \(V_\omega\), that the spectrum of \(H_{\lambda, \omega}\) in

\[
S_{\epsilon}^- = \bigcap_i \left[\nu_i - 2\sqrt{K} + \epsilon, \nu_i + 2\sqrt{K} - \epsilon\right]
\]

is almost surely purely absolutely continuous; the special case \(K = W = 2\) was earlier considered by Froese et al. [6]. Thus we replace the intersection with union (i.e. the fastest Lyapunov exponent with the slowest one) and \(2\sqrt{K}\)