Chaotic Dynamics in an Impact Problem

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Abstract. We consider the model describing the vertical motion of a ball falling with constant acceleration on a wall and elastically reflected. The wall is supposed to move in the vertical direction according to a given periodic function \( f \). We show that a modification of a method of Angenent based on sub- and super-solutions can be applied in order to detect chaotic dynamics. Using the theory of exact symplectic twist maps of the cylinder one can prove the result under “natural” conditions on the function \( f \).

1. Introduction

Mechanical models with impacts appear in many engineering applications, such as the modeling of pneumatic hammers or various machinery with moving parts. This kind of models also appear in Theoretical Physics, for example the Fermi–Ulam oscillator provides a model for the motion of particles between two galaxies. It is not surprising that this topic has been widely studied both from the analytical and numerical point of view. We cite the monograph [3] and the references therein for a good insight.

In this paper we concentrate on a particular impact problem. We consider the model of a free falling ball on a moving racket. The racket is supposed to move periodically in the vertical direction according to a function \( f(t) \) and the ball is reflected according to the law of elastic bouncing when hitting the racket. The only force acting on the ball is the gravity \( g \). Moreover, it is usual to assume that the mass of the racket is huge with respect to the mass of the ball. It means that the impacts do not affect the motion of the racket.

A good strategy to describe the motion of the ball is to define a map \( P \) that sends a couple \((t_0, v_0)\) representing the time of impact and the velocity immediately after it to the next impact time and corresponding velocity \((t_1, v_1)\). In order to write the map, Holmes [6] considered the approximation given by the assumption of a large amplitude of the motion of the ball with respect to the amplitude of the motion of the racket. As an example, he considered the case \( f(t) = \beta \sin \omega t \). In this case the model is the one given by the so-called standard map.
\[
\begin{aligned}
  t_1 &= t_0 + \frac{2}{g} v_0 \\
  v_1 &= v_0 + 2\beta \omega \cos \omega t_1.
\end{aligned}
\]

This map is area preserving and, for \(v_0\) sufficiently large, is defined on the cylinder. It is widely studied and complex dynamics appear when the amplitude \(\beta\) is sufficiently large. Holmes approximation suggests that this very simple mechanical model shows an interesting dynamics.

We are going to study the exact model, without Holmes approximation. In this case one is led to the following map

\[
P: \begin{cases}
  t_1 &= t_0 + \frac{2}{g} v_0 - \frac{2}{g} f[t_1, t_0] + \frac{2}{g} \dot{f}(t_0) \\
  v_1 &= v_0 + \dot{f}(t_1) - 2f[t_1, t_0] + \dot{f}(t_0)
\end{cases}
\]

where

\[
f[t_1, t_0] = \frac{f(t_1) - f(t_0)}{t_1 - t_0}.
\]

This is the map (up to changes of systems of references) considered by Pustyl’nikov [15,16]. He proved that if the motion of the racket is periodic and there exists an instant in which the velocity is sufficiently large, precisely if

\[
\max \dot{f} \geq \frac{g}{2},
\]

then unbounded motions of the ball are possible. Adding some hypothesis on \(f\) he was able to prove that there exists a set of positive measure in the phase-space \((t, v)\) leading to unbounded motions. Nonetheless bounded motions also exist. Actually, in [11] we proved that if \(f\) is regular then for every real number \(\omega\) sufficiently large, there exists a solution with rotation number \(\omega\). Among the huge amount of results concerning such model we cite [4,7,10,13,17]. It is worth mentioning also the paper by Dolgopyat [5] dealing with non-gravitational potentials and the paper by Kunze and Ortega [9] dealing with non-periodic functions \(f\).

Holmes results and the coexistence of bounded and unbounded motions, together with many numerical evidences, motivated us to the analytical study of the chaotic behaviour of the model. Chaotic dynamics are understood as the existence of a compact invariant set \(K\) such that some iterate of the map restricted to it is semi-conjugated to the Bernoulli shift. This is a classical definition of chaos and is used in many contexts. See [8,14] for more details.

On this line, in a recent paper, Ruiz-Herrera and Torres [17] considered the application of a general topological tool based on stretching techniques. In this way, they constructed some particular periodic functions \(f\) for which the corresponding model of bouncing ball shows chaotic dynamics. Note that, as in Holmes approach, their result is valid for a particular choice of the function \(f\). We will show how a different approach, based on symplectic techniques, can give the result for a large class of functions characterized by a “natural” condition.