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The Reduced Phase Space of Palatini–Cartan–Holst Theory

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Abstract. General relativity in four dimensions can be reformulated as a gauge theory, referred to as Palatini–Cartan–Holst theory. This paper describes its reduced phase space using a geometric method due to Kijowski and Tulczyjew and its relation to that of the Einstein–Hilbert approach.

1. Introduction

General relativity (GR) is defined in terms of the metric tensor and of the Einstein–Hilbert (EH) action functional. A (classically) alternative way of formulating it, which has the advantage of being a gauge theory, follows from the observation that the dynamical metric may be expressed in terms of a fixed reference metric via a dynamical (co)frame field [7]. We will refer to this version of GR as Palatini–Cartan–Holst (PCH) theory as detailed below, in a discussion about nomenclature.

The reduced phase space of a theory is the space of possible initial conditions endowed with its natural symplectic structure.[1] For example, in the usual case of mechanics on a target manifold M, it is T^*M with its canonical symplectic structure. In the case of electromagnetism in four dimensions, one starts with a phase space in which the conjugate variables are the vector potential and the electric field on the initial 3-surface with symplectic form induced from their pairing. The reduced phase space is then given by the solution to

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[1] Traditionally, the reduced phase space is defined as the space of solutions endowed with its natural symplectic structure. If the theory is formulated on a manifold of the form $\Sigma \times [a, b]$ and Σ is a Cauchy surface, this is the same as the space of possible initial conditions on Σ. We use a more general definition where Σ is not necessarily Cauchy. In particular, by initial conditions we mean conditions for which there is a, possibly nonunique, local, but not necessarily global, solution.

the Gauss law (vanishing of the divergence of the electric field) modulo gauge transformations. In the case of GR, in space–time dimension greater than two, one starts with a phase space presented as the cotangent bundle of the space of metrics on the initial space-like hypersurface. The reduced phase space is then obtained as the solutions to the so-called energy and momentum constraints modulo diffeomorphisms, both tangential to the space hypersurface and transversal to it.

One well-known method to obtain the reduced phase space, which works well in the above examples, is due to Dirac [14]. The literature is also full of attempts to apply this method to PCH theory.

In this paper, we will study the reduced phase space of PCH theory using instead a geometric method introduced by Kijowski and Tulczyjew [23] (which also has the advantage of being compatible with the BV-BFV formalism introduced in [8,9]). We will show that, under the assumption that the induced boundary metric is nondegenerate, the reduced phase space can be nicely described and corresponds indeed to that of the EH formulation, which has two local degrees of freedom (interpreted as the two possible polarizations of the graviton). Note that this assumption is just an open condition on the space of bulk co-frame fields. We do not compute the reduced phase space without this assumption, but a result proved below suggests that in the case of a light-like boundary the reduction should have no local degrees of freedom.

In a nutshell[2] our result is as follows. We start with the action functional

$$S[e,\omega] = \int_M \mathrm{Tr}\left[e \wedge e \wedge F_\omega + \frac{\Lambda}{4} e^4\right],$$

where M is a four-dimensional manifold with boundary (which admits Lorentzian structures) endowed with a rank-four vector bundle isomorphic to TM with a reference fibre metric, e a tetrad, ω an orthogonal connection and Λ the cosmological constant. We assume that also the boundary restriction of the metric induced by e is nondegenerate. Then our result is that the reduced phase space is obtained by coisotropic reduction in the symplectic space consisting of the space of boundary tetrads, denoted by $\mathbf{e}$, and of boundary connections modulo the action of $e \wedge \cdot$ (Theorem 4.6). Denote by $\boldsymbol{\omega} = [\omega]_e$ the respective equivalence class, the symplectic structure $\omega^\partial = \delta\alpha^\partial$ reads

$$\alpha^\partial = \frac{1}{2} \int_{\partial M} \mathrm{Tr}\left[\mathbf{e} \wedge \mathbf{e} \wedge \delta\boldsymbol{\omega}\right].$$

We show that this reduction is equivalent to the space of boundary tetrads and connections (e, ω) satisfying the structural constraint $p d_\omega e = 0$, where p is the projection (relying on an irrelevant choice of complement) to the space of forms β satisfying $e \wedge \beta = 0$, and using this description we are able to prove that the constraints defining the coisotropic submanifold are

[2] In this Introduction, for simplicity we do not present the extension by Holst depending also on the inner dual of the curvature, which is discussed in details in the rest of the paper.