On Genuine Bernstein–Durrmeyer Operators

Heiner Gonska, Daniela Kacsó, and Ioan Raşa

Dedicated to Academician Dimitrie D. Stancu on the occasion of his 80th birthday

Abstract. We continue the studies on the so-called genuine Bernstein–Durrmeyer operators $U_n$ by establishing a recurrence formula for the moments and by investigating the semigroup $T(t)$ approximated by $U_n$. Moreover, for sufficiently smooth functions the degree of this convergence is estimated. We also determine the eigenstructure of $U_n$, compute the moments of $T(t)$ and establish asymptotic formulas.

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1. Introduction

The present note continues and supplements previous research on the so-called genuine Bernstein–Durrmeyer operators which – according to our present knowledge – were first considered by W. Chen [2] and T. N. T. Goodman and A. Sharma [4] around 1987. They constitute an approximation process for functions $f \in C[0, 1]$ which produces algebraic polynomials and is related to the well-known Bernstein operators $B_n$ as follows. The latter are given by $(f \in C[0, 1], x \in [0, 1], n \in \mathbb{N})$

$$B_n(f; x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k}(x)$$

$$:= \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}.$$
Furthermore, in 1972 A. Lupuş [8] investigated a Beta-type operator defined by
\[
B_n(f; x) := \begin{cases} 
  f(x), & \text{for } x \in \{0, 1\} \\
  \frac{1}{B(nx, n-nx)} \int_0^1 t^{nx-1} (1-t)^{n-1-nx} f(t) dt, & \text{for } x \in (0, 1)
\end{cases}
\]
with \(B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt\), \(a, b > -1\). Both \(B_n\) and \(\overline{B}_n\) are positive linear operators reproducing linear functions and thus interpolating every function \(f \in C[0, 1]\) at 0 and 1. Hence these properties are shared by their composition \(U_n := B_n \circ \overline{B}_n\), the genuine Bernstein–Durrmeyer operators, given explicitly by
\[
U_n(f; x) = f(0) \cdot p_{n,0}(x) + f(1) \cdot p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \int_0^1 p_{n-2,k-1}(t) f(t) dt.
\]
Here \(p_{-1,k} \equiv 0\) and \(p_{0,k} \equiv 1\).

Among the many articles written on the \(U_n\), we mention here only the ones by P. Parvanov and B. Popov [11], by T. Sauer [15], by S. Waldron [16], and the book of R. Păltănea [12]. The reader’s attention is drawn to the fact that the \(U_n\) are sometimes also called “the modified Bernstein–Durrmeyer operators”, “modified Bernstein–Schoenberg operators”, “Bernstein-type operators”, “limit case of Bernstein’s operators with Jacobi weights”, etc. We prefer to use Sauer’s genuine naming.

In the present note Section 2 is devoted to the computation of the moments of \(U_n\) and the images of the monomials under \(U_n\). In Section 3 we investigate the semigroup approximated by \(U_n\) and include a statement for the degree of approximation of sufficiently smooth functions. Section 4 deals with the eigenstructure of \(U_n\), and Section 5 describes the moments of the semigroup approximated by the Bernstein operators (and thus those of the semigroup associated to the \(U_n\)). The final Section 6 provides information concerning the asymptotic behavior of \(U_{2n}\) and some consequences of it.

### 2. The moments of \(U_n\)

First we describe a method to compute recursively the moments of \(U_n\) and the images of the monomials under \(U_n\). Then we establish a recurrence formula for the moments.

Let \(H_{n,x,k}(t) := U_n(e_1 - xe_0)^k(t)\), with \(e_i(t) := t^i\).

**Theorem 2.1.** (i) \(H_{n,x,0}(t) = 1\), \(H_{n,x,1}(t) = t - x\), and for \(k \geq 2\),
\[
t(1-t)H''_{n,x,k} + k(k-1)H_{n,x,k} = k(k-1)[(1-2x)H_{n,x,k-1} + x(1-x)H_{n,x,k-2}].
\]

(ii) Given \(n \geq 1, x \in [0, 1], k \geq 2\), we have
\[
H_{n,x,k}(t) = a_k t^k + \cdots + a_1 t + a_0
\]