Meromorphic Functions That Share One Finite Value CM or IM with Their $k$-th Derivative

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Abstract. In this paper we shall prove that if a non-constant meromorphic $f$ and its $k$-th derivative $f^{(k)}$ ($k \geq 2$) share the value $a \neq 0, \infty$ CM (IM) and if $\tilde{N}(r, \frac{1}{f}) = S(r, f) \left( \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{f_{\tilde{N}}} \right) = S(r, f) \right)$, then $f \equiv f^{(k)}$. These results extend the results in Al-Khaladi (J Al-Anbar Univ Pure Sci 3:69–73, 2009).

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1. Introduction and Results

In this paper the term “meromorphic” will always mean meromorphic in the complex plane. We use the standard notations and results of Nevanlinna theory (see [5] or [6], for example). In particular, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly for a set $E$ of $r$ of finite linear measure. We say that two non-constant meromorphic functions $f$ and $g$ share a value $a$ IM (ignoring multiplicities), if $f$ and $g$ have the same $a$-points. If $f$ and $g$ have the same $a$-points with the same multiplicities, we say that $f$ and $g$ share the value $a$ CM (counting multiplicities). Let $k$ be a positive integer, we denote by $N_k \left( r, \frac{1}{f-a} \right)$ the counting function of $a$-points of $f$ with multiplicity $\leq k$ and by $N_{(k+1)}(r, \frac{1}{f-a})$ the counting function of $a$-points of $f$ with multiplicity $> k$, where each $a$-point is counted according to its multiplicity. Similarly we define $\tilde{N}_k \left( r, \frac{1}{f-a} \right)$ and $\tilde{N}_{(k+1)} \left( r, \frac{1}{f-a} \right)$ where in counting the $a$-points of $f$ we ignore the multiplicities.
In [4] Gundersen proved the following theorem:

**Theorem A.** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share two distinct values \( 0, a \neq \infty \) CM, then \( f \equiv f' \).

In [1] the author considered the case that \( f \) and \( f' \) share only one value and proved the following theorems:

**Theorem B.** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share the value \( a \neq 0, \infty \) CM, and if \( N\left(r, \frac{1}{f}\right) = S(r, f) \), then either \( f \equiv f' \) or

\[
f(z) = \frac{a(z - c)}{1 + Ae^{-z}},
\]

where \( A \neq 0 \) and \( c \) are constants.

**Theorem C.** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f' \) share the value \( a \neq 0, \infty \) IM, and if \( N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f'}\right) = S(r, f) \), then either \( f \equiv f' \) or

\[
f(z) = \frac{2a}{1 - Ae^{-2z}},
\]

where \( A \) is a nonzero constant.

It is asked naturally whether \( f' \) in Theorems B and C can be replaced by \( f^{(k)} \) \((k \geq 2)\). In this paper, we will give a positive answer to this question. Indeed, we shall proved the following theorems:

**Theorem 1.** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f^{(k)} \) \((k \geq 2)\) share the value \( a \neq 0, \infty \) CM, and if \( N\left(r, \frac{1}{f}\right) = S(r, f) \), then \( f \equiv f^{(k)} \).

**Theorem 2.** Let \( f \) be a non-constant meromorphic function. If \( f \) and \( f^{(k)} \) \((k \geq 2)\) share the value \( a \neq 0, \infty \) IM, and if \( N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S(r, f) \), then \( f \equiv f^{(k)} \).

### 2. Lemmas

**Lemma 1.** [2] Let \( k \) be a positive integer, and let \( f \) be a meromorphic function such that \( f^{(k)} \) is not constant. Then either

\[
\left(\frac{f^{(k+1)}}{f^{(k)}}\right)^{k+1} = c\left(\frac{f^{(k)}}{f} - \lambda\right)^{k+2},
\]

for some nonzero constant \( c \), or

\[
kN_{1j}(r, f) \leq N_{(2, f, f^{(k)})}(r, \frac{1}{f^{(k+1)}}) + N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f),
\]

where \( \lambda \) is a constant.