On Strongly Minimal Kähler Surfaces in $\mathbb{C}^3$ and the Equality $scal(p) = 4 \inf sec(\pi r)$

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Abstract. Pursuing an idea motivated by a question of S.-S. Chern from 1968 on the existence of intrinsic Riemannian obstructions to minimality [Chern, S.-S.: Minimal submanifolds in a Riemannian manifold (1968)], an important study of the very idea of curvature was deepened after 1993 by B.-Y. Chen, then by other authors. In the last two decades, B.-Y. Chen’s fundamental inequalities have been investigated by many authors in the context of various geometric structures. In this work, we start by presenting B.-Y. Chen’s fundamental inequality for Kähler submanifolds in complex space forms, and we recall why the case of Kähler surfaces in $\mathbb{C}^3$ satisfying $scal(p) = 4 \inf sec(\pi r)$ appears naturally and is important. Then we provide several characterizations of strongly minimal complex surfaces in the complex three dimensional space. We focus our study on the question of finding further examples of strongly minimal Kähler surfaces, as the question of a complete classification of these geometric objects is still open.

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1. Chen’s Fundamental Inequalities for Complex Submanifolds

1.1. Motivation of Our Study

B.-Y. Chen’s recent monograph [24] summarizes and surveys the important developments obtained in the last two decades in the study of new curvature invariants, called by many authors Chen invariants; the corresponding inequalities these curvature invariants satisfy are called Chen’s inequalities; for terminology and various facets of this important class of problems see e.g. [2–11,14–17,19,26–34,36,37,39–43,46–51,53–58,60]. At page 252 in [24],
it is pointed out that the motivation to study the new curvature invariants originates in S.-S. Chern question from [25] to search for further Riemannian obstructions for a Riemannian manifold to admit an isometric minimal immersion into a Euclidean space. As L. Verstraelen points out in the Foreword to [24], the value of this monograph resides in deepening the “understanding of the concepts and theories under discussion”, “as well as for their relevance in other sciences and in philosophy”.

From among the topics covered in [24], we focus on section 15.5, titled Examples of strongly minimal Kähler submanifolds. I. Dimitrić is one of the authors who concluded [35] that “it is important to understand strongly minimal Kähler surfaces and provide additional examples of them.”

To every class of fundamental inequalities obtained by pursuing the quest of a best possible immersion into a given ambient space it corresponds a class of geometric objects satisfying the equality case. In some cases, these objects can be characterized or classified. The strongly minimal Kähler surfaces are related to the equality \(\text{scal}(p) = 4 \inf \text{sec}(\pi^r)\), as we remind in detail below.

As section 15.5 from [24] shows, there are very few known examples of strongly minimal Kähler submanifolds. Why is this the case? Why is it challenging to obtain further examples? The present work describes the quest for strongly minimal submanifolds and why this quest is important.

1.2. B.-Y. Chen’s Kählerian Curvature Invariants

Let \(M^n\) be a Kähler manifold of complex dimension \(n\). Let us denote by \(J\) its complex structure.

Let \(\pi \subset T_pM\) be a plane section. Then \(\pi\) is called totally real if \(J\pi\) is perpendicular to \(\pi\). We denote by \(\text{sec}(\pi^r)\) the sectional curvature of a totally real plane section \(\pi^r\). Denote

\[
(\inf \text{sec}^r)(p) = \inf_{\pi^r \in T_pM} \text{sec}(\pi^r),
\]

where \(\text{sec}(\pi^r)\) runs over all totally real plane sections at \(p \in M\).

We denote by \(\text{sec}(X \wedge Y)\) and \(\text{scal}(p)\) the sectional curvature of the plane determined by the vectors \(X\) and \(Y\) and respectively the scalar curvature at the point \(p\). A classical result (see [59, p. 820]) states that the maximal sectional curvature (in absolute value) occurs in a holomorphic direction. Also (see [24, p. 318]) it is known that for a Kähler submanifold \(N\) with complex dimension \(\geq 2\) in a complex space form \(M^m(4c)\), we have \(\inf \text{sec}^r \leq c\), with the equality holding identically if and only if \(N\) is a totally geodesic Kähler submanifold.

Consider \(U\) a coordinate chart on \(M\) and \(e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n\) a local orthonormal frame on \(U\). Then we have at \(p \in U\):

\[
\text{scal}(p) = \sum_{i<j} \text{sec}(e_i \wedge e_j), \quad i, j = 1, \ldots, n, 1^*, \ldots, n^*. \tag{1}
\]

For each real number \(k\), B.-Y. Chen’s Kählerian invariant of order 2 and coefficient \(k\) at \(p \in M\) is defined [20] by