Solutions on Asymptotically Periodic Elliptic System with New Conditions

Dongdong Qin and Xianhua Tang

Abstract. This paper is concerned with the following elliptic system:

\[
\begin{align*}
-\Delta u + U_1(x)u &= F_u(x, u, v) \quad \text{in } \mathbb{R}^N, \\
-\Delta v + U_2(x)v &= F_v(x, u, v) \quad \text{in } \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N).
\end{align*}
\]

Assuming that the potential \(U_i(x)\) are periodic in \(x\) and 0 lies in a spectral gap of \(\sigma(-\Delta + U_i), i = 1, 2\), two types of ground state solutions are obtained with some new super-quadratic conditions on nonlinearity \(F\) which are weaker than some well known ones. For the case that \(U_i(x)\) and \(F(x, u, v)\) are asymptotically periodic in \(x\), a nontrivial solution is established by using a generalized linking theorem and some new techniques.

Mathematics Subject Classification. 35J20, 35J50.

Keywords. Elliptic system, superlinear, asymptotically periodic, ground state, strongly indefinite functionals.

1. Introduction and Main Results

Consider the following elliptic system:

\[
\begin{align*}
-\Delta u + U_1(x)u &= F_u(x, u, v) \quad \text{in } \mathbb{R}^N, \\
-\Delta v + U_2(x)v &= F_v(x, u, v) \quad \text{in } \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N),
\end{align*}
\]

where \(U_1, U_2 \in C(\mathbb{R}^N, \mathbb{R})\) and \(F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})\) with gradient \(\nabla F = (F_u, F_v)\). This problem arises in applications from mathematical physics, and solutions of \((1.1)\) can be interpreted as stationary states of the corresponding reaction–diffusion system which models phenomena from chemical and biological dynamics. Problem \((1.1)\) has been extensively investigated in the
literature based on various assumptions on the potential $U_i(x)$ and the nonlinearity $F(x, z)$ with $z = (u, v) \in \mathbb{R}^2$.

For the case of a bounded domain, assuming moreover $U_1(x) \equiv \xi$ and $U_2(x) \equiv \eta$, (1.1) reduces to the following elliptic system:

$$
\begin{aligned}
-\Delta u + \xi u &= F_u(x, u, v) \quad \text{in } \Omega, \\
-\Delta v + \eta v &= F_v(x, u, v) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
$$

(1.2)

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. System (1.2) is called resonant if $\{-\xi, -\eta\} \cap \sigma(-\Delta) \neq \emptyset$, otherwise it is nonresonant, where $\sigma(-\Delta) = \{\lambda_k : 0 < \lambda_1 < \lambda_2 < \cdots, k = 1, 2, \ldots\}$ denotes the eigenvalues of the Laplacian operator on $\Omega$ with zero boundary condition. A vast literature on the study of the existence and multiplicity of solutions for elliptic systems like or similar to (1.2) via the critical point theory has grown since Costa and Magalhães [7] published their paper, see [4, 7, 20, 23, 30, 33, 45] and the references therein. In [7], a unified approach to both the cooperative and noncooperative systems is introduced, and some existence results to (1.2) for both resonant and nonresonant cases were obtained by minimax techniques under a condition which was called nonquadraticity at infinity. Li and Liu [20] established two nontrivial solutions by taking advantage of saddle point reduction. Relying on computations of critical groups and the Morse theory, the existence and multiplicity of solutions to system (1.2) was established by Ma [30], Lu and Su [29], see also [33] where a penalization technique was used. In [45], using the methods used in [18], infinitely many solutions was obtained under the oddness and boundedness assumptions on the nonlinearity. In recent paper [4], relying on two variant fountain theorems developed by Zou [47], Chen and Ma obtained infinitely many solutions of (1.2) with sublinear or superlinear terms. Later, this results was improved by Li and Tang [23] by using the minimax methods.

For related topics, we refer readers to [1, 8–10, 46] and the references therein.

Here, we mention the recent work of Chen and Ma [5]. Based on a generalized Nehari manifold method developed by Szulkin and Weth [37, 38], ground state solution of Nehari–Pankov type to system (1.2) was obtained by Ma [30], Lu and Su [29], see also [33] where a penalization technique was used. In [45], using the methods used in [18], infinitely many solutions was obtained under the oddness and boundedness assumptions on the nonlinearity. In recent paper [4], relying on two variant fountain theorems developed by Zou [47], Chen and Ma obtained infinitely many solutions of (1.2) with sublinear or superlinear terms. Later, this results was improved by Li and Tang [23] by using the minimax methods. For related topics, we refer readers to [1, 8–10, 46] and the references therein.

Here, we mention the recent work of Chen and Ma [5]. Based on a generalized Nehari manifold method developed by Szulkin and Weth [37, 38], ground state solution of Nehari–Pankov type to system (1.2) was obtained in [5], i.e. a nontrivial solution $z_0$ which satisfies $\Phi(z_0) = \inf_{N^-} \Phi$, where the Nehari–Pankov manifold $N^-$ is defined by

$$
N^- = \{z \in E|E^- : \langle \Phi'(z), z \rangle = \langle \Phi'(z), \zeta \rangle = 0, \forall \zeta \in E^-\},
$$

(1.3)

$\Phi$ is the energy functional and $E = E^- \oplus E^+$ is a Hilbert space on which $\Phi$ defines. The generalized Nehari manifold method used there [5] is based on a direct and simple reduction of the strongly indefinite problem to a definite one. More precisely, a homeomorphism between the Nehari–Pankov manifold $N^-$ and a unit sphere in $E^+$ is established which allows to find a minimizing sequence on the sphere and hence on the Nehari–Pankov manifold (cf. [37, 38]). In recent paper [40, 42], Tang introduced a Non-Nehari manifold method which