On the Composition and Decomposition of Positive Linear Operators (V)

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Abstract. The present note supplements information contained in three earlier papers with the same title. Here we consider certain aspects of (in)decomposability of positive linear operators given on $C(X)$ where $X$ is a compact convex metrizable subset of a topological vector space which is locally convex and Hausdorff.

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1. Introduction

Over the recent years there has been continuing interest in the composition and decomposition of positive linear operators. Both operations mentioned can be useful to foster progress in Approximation Theory.

As an example for the usefulness of composition we mention compositions of type $C \circ S_{\Delta_n}$ where $C$ is a convolution-type operator and $S_{\Delta_n}$ is piecewise linear interpolation on a sequence $a = x_0 < x_1 < \cdots < x_n = b$ in the compact interval $[a, b]$. This approach was used to create discrete polynomial operators solving various forms of a problem posed by Butzer in 1980. See, for example, the paper by Gavrea et al. [12] and the references cited there.

Composition can also be misused. The product $P = \prod_{i=1}^{n} P_i$ of $n$ positive linear operators is again a positive linear operator. Thus an uncountable zoo

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of operators $P$ can be constructed based on previously known mappings $P_i$. The drawback is that the second moments of the individual mappings add up following a certain pattern (see [14]) and that the product operators become more and more complicated. The reader should consult [14] for an example.

In many of the composition/decomposition papers the classical Bernstein operator $B_n : C[0, 1] \to \pi_n$ played a relevant role, and that in combination with Lupaş’ Beta-operator of the second kind. As an example we mention the facts that the genuine Bernstein–Durrmeyer operators $U_n$ have the decomposition $U_n = B_n \circ \overline{B}_n$, while $S_n = \overline{B}_n \circ B_n$ is a special Stancu operator (see [20]).

The question arose if $B_n$ itself can be decomposed into even ‘simpler’ building blocks $P$ and $Q$, $B_n = P \circ Q$. The first attempt made in this direction was carried out by Gonska and Lupaş who defined

$$G_n := \overline{B}_n \circ S_n$$

where $S_n$ is piecewise linear interpolation at $0 = x_0 < x_1 = \frac{1}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1$.

However, in [15] it was shown that $G_2 \neq B_2$, that there is also no polynomial operator $Q$ giving $B_n = \overline{B}_n \circ Q$, and that it is impossible to write $B_n = L \circ S_{\Delta_n}$ for a large class of positive integral operators $L$. We cite from [15]: ‘... these negative results do not exclude the possibility that there are non-trivial decompositions $B_n = P \circ Q$ with $P \neq \overline{B}_n$ or $Q \neq S_{\Delta_n}$. But if one insists in the choice $P = \overline{B}_n$, then we are necessarily led to certain non-positive operators $F_n \ldots$’ replacing $Q$.

This operator $F_n = \overline{B}_n^{-1} \circ B_n$ was further discussed in [15,18] and [19]. It has many interesting properties, but several questions (such as convergence on all of $C[0, 1]$) remain open.

The present note can be considered as a continuation of our paper [17] where we dealt with the univariate case. More specifically, here we deal with the question of indecomposability/decomposability of the forms $B = L \circ K$ and $L = B \circ K$ where $L$ stands for certain positive linear operators, $B$ mimicks the Bernstein operator and $K$ replaces piecewise linear interpolation. All this is done for functions on $C(X)$ where $X$ is a compact and convex metrizable subset of a locally convex Hausdorff real space, and the aim is again to show that certain compositions can only be trivial.

After some preliminary results in Sect. 2, the next section deals with the case $B = L \circ K$. Section 4 finally treats $L = B \circ K$.

2. Preliminary Results

For the definitions and notations used in this section see, e.g., [2, Chap. 1], or [3, Chap. 1].

Let $X$ be a compact convex metrizable subset of a locally convex Hausdorff real space and $M^+(X)$ the cone of all positive Radon measures on $X$. 