Packing 4-Cycles in Eulerian and Bipartite Eulerian Tournaments with an Application to Distances in Interchange Graphs

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Abstract. We prove that every Eulerian orientation of $K_{m,n}$ contains $\frac{1}{4+\sqrt{8}}mn(1-o(1))$ arc-disjoint directed 4-cycles, improving earlier lower bounds. Combined with a probabilistic argument, this result is used to prove that every regular tournament with $n$ vertices contains $\frac{1}{8+\sqrt{32}}n^2(1-o(1))$ arc-disjoint directed 4-cycles. The result is also used to provide an upper bound for the distance between two antipodal vertices in interchange graphs.

Keywords: tournaments, packing, cycles, interchange graphs

1. Introduction

All graphs and digraphs considered here are finite, and contain no parallel edges or antiparallel arcs. For standard graph-theoretic terminology the reader is referred to [2]. An Eulerian orientation of an undirected graph $G$ is an orientation of the edges of $G$ such that each vertex has the same indegree and outdegree in the resulting oriented graph. It is well known that a graph $G$ has an Eulerian orientation if and only if every vertex is of even degree. A tournament is an orientation of a complete graph and a bipartite tournament is an orientation of a complete bipartite graph. Properties of Eulerian tournaments (also called regular tournaments) and bipartite Eulerian tournaments have been extensively studied in the literature (see, e.g., [3, 8, 9]).

A conjecture of Brualdi and Shen [5] asserts that every bipartite Eulerian tournament has a decomposition into $C_4$, the directed cycle on four vertices. In other words, any Eulerian orientation of $K_{m,n}$ contains $mn/4$ arc-disjoint $C_4$’s (clearly $m$ and $n$ must be even in order to have an Eulerian orientation). This conjecture is still far from being solved, and, in fact, there is no evidence that it should be true. Even an asymptotic version of this conjecture is not known. A recent result from [11] states that any Eulerian orientation of a bipartite graph with vertex class sizes $m$ and $n$ and more than $2mn/3$ arcs has a $C_4$. This immediately implies that any Eulerian orientation of $K_{m,n}$ contains
at least $mn/12$ arc-disjoint $C_4$’s. Although very far from the Brualdi-Shen conjecture, this is currently the best known published lower bound for this problem. In this paper we considerably improve the lower bound.

**Theorem 1.1.** Every Eulerian orientation of $K_{m,n}$ contains at least $\frac{1}{4+\sqrt{8}}mn(1-o(1))$ arc-disjoint copies of $C_4$.

The $o(1)$ term denotes a function tending to zero as $\min\{m, n\}$ tends to infinity. Notice that, trivially, any $C_4$-packing of an orientation of $K_{m,n}$ contains at most $mn/4$ elements. Theorem 1.1 shows that more than 58 per cent of the arcs can be packed with $C_4$’s.

The motivation behind the Brualdi-Shen conjecture stems from the theory of interchanges. These graphs are defined as follows: Let $R = (r_1, \ldots, r_n)$ and $S = (s_1, \ldots, s_n)$ be non-negative integral vectors with $\sum r_i = \sum s_j$. Let $A(R, S)$ denote the set of all $m \times n \{0, 1\}$-matrices with row sum vector $R$ and column sum vector $S$, and assume that $A(R, S) \neq \emptyset$. This set has been studied extensively (see [4] for a survey). An interchange is a transformation which replaces a $2 \times 2$ identity submatrix of a matrix $A$ with the $2 \times 2$ permutation matrix $P_{i,j} = |i - j|$. Clearly an interchange (and hence any sequence of interchanges) does not alter the row and column sum vectors of a matrix and transforms a matrix in $A(R, S)$ into another matrix in $A(R, S)$. Ryser [10] proved that for any two matrices in $A(R, S)$ there is a sequence of interchanges which transforms one to the other. The interchange graph $G(R, S)$ of $A(R, S)$, defined by Brualdi in 1980, is the graph with all matrices in $A(R, S)$ as its vertices, where two matrices are adjacent provided that one can be obtained from the other by a single interchange. Brualdi conjectured that the diameter of $G(R, S)$, denoted $d(R, S)$, cannot exceed $mn/4$. This conjecture is still far from being resolved. The best known bounds for $d(R, S)$ use the following reduction of Walkup. Given two matrices $A, B \in A(R, S)$ consider the $m \times n$ matrix $A - B$. This matrix contains the elements $\{-1, 0, 1\}$ and the sum of each row and column is zero. Thus, there is a one-to-one correspondence between such matrices and Eulerian orientations of a bipartite graph with vertex class sizes $m$ and $n$. Walkup [12] proved that the distance between $A$ and $B$ in $G(R, S)$, denoted $i(A, B)$ satisfies

\[
i(A, B) = \frac{d(A, B)}{2} - q(A, B)\]  

(1.1)

where $d(A, B)$ is the number of nonzero entries in $A - B$ and $q(A, B)$ is the maximum number of arc-disjoint cycles in a cycle decomposition of the bipartite Eulerian digraph corresponding to $A - B$. The currently best known upper bound for $d(R, S)$ obtained using Walkup’s reduction is given in [11] yielding $d(R, S) \leq \frac{m}{2}$. We call two matrices $A, B \in A(R, S)$ antipodal if $B$ is the complement of $A$, or, in other words, if $A + B$ is the all-one matrix. Notice that a necessary condition for $A$ and $B$ to be antipodal is that all coordinates of $R$ are $m/2$ and all coordinates of $S$ are $n/2$. Trivially, the distance between two antipodal vertices in $G(R, S)$ is at least $mn/4$, and hence such vertices show that Brualdi’s conjecture, if true, is best possible. In particular antipodal vertices are conjectured to be farthest apart in $G(R, S)$. If $A$ and $B$ are antipodal matrices then $A - B$ corresponds to an Eulerian orientation of $K_{m,n}$. Observe that using (1.1) and Theorem 1.1 we obtain the following.