Freely Braided Elements in Coxeter Groups

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Abstract. We introduce a notion of “freely braided element” for simply laced Coxeter groups. We show that an arbitrary group element \(w\) has at most \(2^{N(w)}\) commutation classes of reduced expressions, where \(N(w)\) is a certain statistic defined in terms of the positive roots made negative by \(w\). This bound is achieved if \(w\) is freely braided. In the type \(A\) setting, we show that the bound is achieved only for freely braided \(w\).

Keywords: braid relation, commutation class, Coxeter group, root sequence

1. Introduction

A well-known result in the theory of Coxeter groups states that any two reduced expressions for the same element \(w\) of a Coxeter group are equivalent under the equivalence relation generated by braid relations. If \(w\) is such that any two of its reduced expressions are equivalent by short braid relations (i.e., iterated commutations of commuting generators), we call \(w\) “fully commutative,” following Stembridge [8].

In this paper, we introduce and study “freely braided elements” for an arbitrary simply laced Coxeter group. This is a more general class of elements than the aforementioned fully commutative elements. The idea behind the definition is that although it may be necessary to use long braid relations in order to pass between two reduced expressions for a freely braided element, the necessary long braid relations in a certain sense do not interfere with one another.

Every reduced expression for a Coxeter group element \(w\) determines a total ordering of the set of positive roots made negative by \(w\). The resulting sequences, called root sequences, play a central role in this paper. We are particularly interested in triples of roots of the form \(\{\alpha, \alpha + \beta, \beta\}\), where each root in the triple is made negative by \(w\) (inversion triples). An inversion triple that occurs consecutively in some root sequence for \(w\) will be called contractible, and the statistic \(N(w)\) mentioned in the abstract above is the number of contractible inversion triples of \(w\). We note that Fan and Stembridge
have already shown in [5, Theorem 2.4] that an element of a simply laced Coxeter group is fully commutative if and only if it has no inversion triples.

In §4 of this paper, we prove that the number of commutation classes (short braid equivalence classes of reduced expressions) of \( w \) is bounded above by \( 2^{N(w)} \), and this bound is achieved if \( w \) is freely braided. Furthermore, a freely braided element \( w \) has a root sequence in which each contractible inversion triple occurs as a consecutive subword (Theorem 4.2.3). In §5, we prove that if the Coxeter group is of type \( A \), then the \( 2^{N(w)} \) bound is achieved only for freely braided \( w \). Our proof relies on the fact that every inversion triple is contractible in the type \( A \) setting.

Along the way, we give a short proof that the commutation graph of any element of a simply laced Coxeter group is bipartite, extending a result in [4].

Another possible approach to proving these results is to use Viennot’s heaps of pieces [10]. These are certain labelled posets that can be associated to commutation classes of reduced expressions for elements of a Coxeter group (see [8, §1.2]). It can be shown that the dual of the heap of a reduced expression is isomorphic, as an abstract poset, to a certain poset arising naturally from the associated root sequence. However, we find it more convenient to argue directly with root sequences in this paper.

2. Preliminaries

2.1. Basic Terminology and Notation

Let \( (W, S) \) be a Coxeter system, with Coxeter matrix \( \{m(s, t)\}_s,t \in S \). It will be assumed throughout this paper that \( (W, S) \) is simply laced, so that \( m(s, t) \in \{2, 3\} \) for all pairs of distinct \( s, t \in S \). We refer to \( W \) itself as a “Coxeter group”. The basic facts concerning Coxeter systems can be found in [3] or [6].

Let \( S^* \) be the free monoid generated by \( S \). The elements of \( S^* \) will be written as finite sequences, e.g., \( s = (s_1, \ldots, s_n) \). We define the length of any \( s \in S^* \) to be the number of its entries.

There is a natural morphism of monoids \( \phi: S^* \longrightarrow W \), given by \( \phi(s_1, \ldots, s_n) = s_1 \cdots s_n \) (the empty sequence is mapped to the identity, \( e \)). We say that an element \( s \in S^* \) represents its image \( w = \phi(s) \in W \); furthermore, if the length of \( s \) is minimal among the lengths of all the sequences that represent \( w \), then we say that \( s \) is reduced, and we call \( s \) a reduced expression for \( w \). The length of \( w \), denoted by \( \ell(w) \), is then equal to the length of \( s \).

Let \( V \) be a vector space over \( \mathbb{R} \) with basis \( \{\alpha_s: s \in S\} \), and let \( B \) be the Coxeter form on \( V \) associated to \( (W, S) \). This is the symmetric bilinear form satisfying \( B(\alpha_s, \alpha_t) = -\cos \frac{\pi m(s, t)}{m(s, t)} \) for all \( s, t \in S \). We shall view \( V \) as the underlying space of a reflection representation of \( W \), determined by the equalities \( \alpha_s \alpha_t = \alpha_t \alpha_s - 2B(\alpha_s, \alpha_t)\alpha_s \), for \( s, t \in S \)

The Coxeter form is preserved by \( W \) relative to this representation.

Define \( \Phi = \{w\alpha_s: w \in W \text{ and } s \in S\} \). This is the root system of \( (W, S) \). Let \( \Phi^+ \) be the set of all \( \beta \in \Phi \) such that \( \beta \) is expressible as a linear combination of the \( \alpha_s \) with nonnegative coefficients, and let \( \Phi^- = -\Phi^+ \). Then \( \Phi \) equals the disjoint union \( \Phi^+ \cup \Phi^- \) [6, §5.4]. The elements of \( \Phi^+ \) (respectively, \( \Phi^- \)) are called “positive” (respectively, “negative”) roots. The \( \alpha_s \) are often referred to as “simple” roots.