Forrester’s Conjectured Constant Term Identity II

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Abstract. We continue our study on Forrester’s conjectured constant term identity which is
equivalent to a new kind of generalization of the Selberg integral. The special cases \( N_1 = 2, 3 \) of
the conjecture have been verified in our previous paper [6]. We show the conjecture holds in the
other extreme case \( N_1 = N - 1 \). The proof is based on the integration formula of Jack polynomials
and the Chu-Vandermonde formula for the generalized binomial coefficients.

Keywords: Morris constant term identity, Selberg integral, Jack polynomial, Chu-Vandermonde
formula

1. Introduction

Suppose \( a, b, k \in \mathbb{Z}_{\geq 0} \) and let

\[
H(x_1, \ldots, x_N; a, b, k) := \prod_{l=1}^{N} (1 - x_l)^a \left( 1 - \frac{1}{x_l} \right)^b \prod_{i \neq j}^{N} \left( 1 - \frac{x_i}{x_j} \right)^k.
\]

The Morris constant term identity [9] states that

\[
CT_{\{x\}} H(x_1, \ldots, x_N; a, b, k) = M_N(a, b, k),
\]

where \( CT_{\{x\}} \) denotes the constant term in the Laurent polynomial and

\[
M_N(a, b, k) := \prod_{l=0}^{N-1} \frac{\Gamma(a + b + 1 + kl)\Gamma(1 + k(l + 1))}{\Gamma(a + 1 + kl)\Gamma(b + 1 + kl)\Gamma(1 + k)}.
\]

In [3], after some numerical computations, Forrester suggested the following remarkable
generalization of this identity and proved the special cases \( a = b = 0 \) (general \( k, N_0, N_1 \)) and \( k = 1 \) (general \( a, b, N_0, N_1 \)).
Conjecture 1.1. We have
\[
CT_{(a)} \prod_{i,j=N_0+1,i\neq j}^{N} \left(1 - \frac{x_i}{x_j}\right) H(x_1, \ldots, x_N; a, b, k)
\]
\[
= M_{N_0}(a, b, k) \prod_{j=0}^{N-1} \frac{(j+1)\Gamma((k+1)(j+a+b+kN_0+1))\Gamma((k+1)(j+1)+kN_0)}{\Gamma(k+1)\Gamma((k+1)(j+a+kN_0+1))\Gamma((k+1)(j+b+kN_0+1))},
\]
where \(N = N_0 + N_1\).

The purpose of this paper is to show

Theorem 1.2. The conjecture holds in the extreme case \(N_1 = N - 1\).

In our previous paper [6] we verified the other extreme cases of \(N_1 = 2, 3\) by invoking the integration formula of Jack polynomials \([4, 5]\) and our proof here also relies upon this formula. We shall also make use of the Chu-Vandermonde formula for the generalized binomial coefficients \([7]\).

It is known \([3]\) that the above conjecture is equivalent to the following integral evaluation.

\[
\int_{[0, 1]^N} \prod_{i=1}^{N} t_i^{N_i - 1} (1 - t_i)^{-1} \prod_{j=N_0+1}^{N} r_j^{N_j-1} \prod_{1\leq i < j \leq N} (t_i - t_j)^2 \prod_{1 \leq j \leq N} (t_i - t_j)^{2k} dt_1 \cdots dt_N
\]
\[
= \prod_{j=0}^{N-1} \frac{\Gamma(y + kl)\Gamma(x + k(N - 1 - l))\Gamma(1 + l + 1)}{\Gamma(x + y + k(N - 1 + 1))\Gamma(1 + k)}
\]
\[
\times \prod_{j=0}^{N-1} \left[ \frac{(j+1)\Gamma(y + (k+1)(j+kN_0))\Gamma(x + k(N - 1 - N_0 - j) - j)}{\Gamma(x + y + k(N - 1 + N_0 + j) + j)} \right].
\]

(1.1)

So the conjecture can be considered as a new kind of generalization of the celebrated Selberg integral \([1, 10]\). This is clearly equivalent to the symmetrized form:

\[
\int_{[0, 1]^N} \prod_{i=1}^{N} t_i^{N_i - N_1} (1 - t_i)^{-1} f_{N,N_0}(t_1, \ldots, t_N) \prod_{1 \leq i < j \leq N} (t_i - t_j)^{2k} dt_1 \cdots dt_N
\]
\[
= \frac{N!}{N_0!N_1!} \text{RHS of (1.1)},
\]

(1.2)

where

\[
f_{N,N_0}(t_1, \ldots, t_N) := \sum_{1 \leq i_1 < \cdots < i_{N_0} \leq N} (t_{i_1} \cdots t_{i_{N_0}})^{N_1-1} \prod_{j_1 < j_2} (t_{j_1} - t_{j_2})^{2k}.
\]

We shall prove this form of the conjecture in the case \(N_1 = N - 1\).

In passing we note that there has been given a \(q\)-analogue of the conjecture by Baker and Forrester in \([2]\).