Exponential attractors for a class of reaction-diffusion problems with time delays

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Abstract. We consider a reaction-diffusion system subject to homogeneous Neumann boundary conditions on a given bounded domain. The reaction term depends on the population densities as well as on their past histories in a very general way. This class of systems is widely used in population dynamics modelling. Due to its generality, the longtime behavior of the solutions can display a certain complexity. Here we prove a qualitative result which can be considered as a common denominator of a large family of specific models. More precisely, we demonstrate the existence of an exponential attractor, provided that a bounded invariant region exists and the past history decays exponentially fast. This result will be achieved by means of a suitable adaptation of the $\ell$-trajectory method coming back to the seminal paper of Málek and Nečas.

1. Introduction

A large number of mathematical models in population dynamics have the following form (see, e.g., [1, 2, 3, 12, 18, 19, 20, 21, 26, 37, 38, 42, 43, 44, 45] and references therein)

$$\frac{\partial u}{\partial t} - D \Delta u = F(u, u^t), \quad \text{in } \Omega \times (0, \infty),$$

where $\Omega$ is a bounded, open and connected subset of $\mathbb{R}^h, h \in \mathbb{N}$. Here $u = (u^1, \ldots, u^M) : \Omega \times \mathbb{R} \to [0, \infty)^M$ represents the population density vector and $D = \text{diag}[d_1, \ldots, d_M]$ is a diffusion matrix ($d_i > 0$, $i = 1, \ldots, M$). Moreover, $F$ is a reaction function which depends not only on $u(x, t)$, but also includes a general functional dependence on the past history up to $t$, denoted by

$$u^t(\cdot, s) = u(\cdot, t+s), \quad s \in (-\infty, 0].$$

We suppose that the populations are isolated, so that system (1.1) is subject to the Neumann homogeneous boundary conditions

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, \infty).$$

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In addition, in view of (1.2), the initial datum is the past history up to \( t = 0 \), i.e.,
\[
\mathbf{u}^0 = \eta, \quad \text{in } \Omega \times (-\infty, 0].
\]  
(1.4)

Among the main goals in the investigation of such kind of systems there are, in particular, the stability of the equilibria and the existence of (stable) periodic solutions. However, if \( M \) is very large even the determination of the equilibria might be an extremely difficult task. Therefore, dealing with very general situations, it seems convenient to interpret the model as a dissipative dynamical system in a suitable phase space, trying to prove the existence of sufficiently small (i.e., finite fractal dimensional) compact invariant sets which characterize the longtime dynamics of the model itself. Typical mathematical objects possessing the mentioned features are the global attractor (see, e.g., [9, 23, 40]) and the exponential attractor (cf. [13, 14, 15]). It is also worth seeing [31] for an updated review of such notions.

The key step of such an analysis is a suitable choice of the underlying phase space (cf. the pioneering contribution [24]). One possibility is to introduce a new variable, the so-called summed past history, which accounts for the memory effects, and obeys certain first order dissipative equation (see [22], or [6, 7] for equations like (1.1)). This approach is useful when one wants to analyze the stability of the attractor with respect to certain parameters of the given system (e.g., in [6], the relaxation times). The limitation of the summed past history approach is that the delay has to be described in terms of a convolution with rather restricted class of kernels.

Another approach is the construction of the so-called trajectory attractor (see [9], cf. also [8].) Here the phase space consists of negative semi-trajectories and the dynamical system is defined by means of a translation semigroup. Note that this is a natural setting for the problems with delay, and no additional variables are needed. A detailed comparison of the past history approach and the concept of trajectory attractor is given in [8]. It turns out that if both methods are applicable, they are equivalent regarding the notion of global attractors, but the advantages of the former stand out.

Nonetheless, there are models to which the past history approach does not apply, while the trajectory one does. For instance, in the case of discrete state-dependent delays, a recent result about the existence of the global trajectory attractor is proven in [36] (see also references therein). There the author also treats the case of distributed (but finite) delays by means of a more conventional approach which regards the delays as additional variables. In the case of infinite delays, a meaningful example can be easily given by exploiting the generality of (1.1). Indeed, hereditary effects can be modelled, for instance, as follows (see, e.g., [39])

\[
\int_{-\infty}^{0} K(\mathbf{u}(t+s), s) ds.
\]

In the present paper, we want to cover such general classes of models; hence the summed past history approach is not applicable. As a phase space, we take the set of the histories \( \eta \),