On decay estimates

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Abstract. We show here that decay estimates can be derived simply by integral inequalities. This result allows us to prove these kind of estimates, with a unified proof, for different nonlinear problems, thus obtaining both well known results (for example for the p-Laplacian equation and the porous medium equation) and new decay estimates.

1. Introduction and statement of results

It is well known that the solution of the linear heat equation

\[ u_t = \Delta u, \quad \mathbb{R}^N \times (0, +\infty) \]

with initial datum \( u_0 = u(x, 0) \in L^{r_0} (\mathbb{R}^N), \ r_0 \geq 1, \) satisfies

\[ \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C \frac{\|u_0\|_{L^{r_0}(\mathbb{R}^N)}}{t^{N/r_0}}, \quad t > 0. \] (1.1)

The previous estimate remains true, also for the solution of the heat equation in a generic open set \( \Omega \) of \( \mathbb{R}^N \) with zero boundary condition

\[
\begin{align*}
& u_t - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \\
& u = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
& u(x, 0) = u_0(x) \quad \text{on } \Omega.
\end{align*}
\]

Moreover, if the measure of \( \Omega \) (denoted by \( |\Omega| \)) is finite, the following exponential decay holds

\[ \|u(t)\|_{L^\infty(\Omega)} \leq C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{N/r_0} e^{\sigma t}}, \quad \forall t > 0. \] (1.2)

The bound (1.1), or more in general estimates of the kind

\[ \|u(t)\|_{L^\infty(\Omega)} \leq C \frac{\|u_0\|_{L^{r_0}(\Omega)}}{t^{h_0} e^{h_1 t}}, \quad t > 0, \] (1.3)

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where $\Omega$ is an open set of $\mathbb{R}^N$, $r_0 \geq 1$, $h_0$ and $h_1$ are suitable positive constants and, as before, $u_0$ is the initial datum $u(x, 0)$, are known in literature as decay estimates or ultracontractive estimates (following the terminology of [16]).

These estimates have been proved not only for the heat equation but also for very different problems, linear or nonlinear (or even doubly nonlinear), degenerate or singular like, for example, the p-Laplacian equation, the porous media equation, the fast diffusion equation, the doubly nonlinear equation etc. (see for example [1,3,4,6,7,9,12,14,16,17,19,20,22,23,27,28,30,32] and the references cited therein).

The importance of these estimates is clear since they show a “strong regularizing” effect, that is, using the words of Brezis and Crandall : “for every $t > 0$ we have $u(t) \in L^{\infty}(\Omega)$ if only $u_0 \in L^1(\Omega)$” (see [13] where, among the results, there is an interesting use of these estimates to get uniqueness results).

Moreover, these estimates describe the behaviour of the solution both for “t large” (i.e. how the solution decays when $t$ tends to $+\infty$) and for “t small” (i.e. what happens to the solution when $t$ tends to zero).

The proofs of these estimates vary from problem to problem. In most cases suitable families of logarithmic Sobolev inequalities, which reflect the operator involved in the problem, are derived. These inequalities are similar to the well-known Gross’logarithmic Sobolev inequalities (see [21] and also [15]).

The aim of this paper is to show that these decay or ultracontractive estimates can be derived simply by integral inequalities (see Theorem 2.1 in Sect. 2 below). More in detail, if a function $u$ (even if it doesn’t solve any problem) satisfies an energy type estimate where $u_0$ is not involved and a further integral inequality involving $u_0$, then $u$ satisfies the decay estimate (1.3) with $h_0$ and $h_1$ given explicitly in terms of the exponents that appear in the energy estimate, on the dimension $N$ and on $r_0$.

Moreover, if the previous integral inequalities are satisfied and $\Omega$ has a finite measure, for a suitable choice of the coefficients involved in the energy estimate we can derive “faster decay estimates” like exponential decay

$$
\|u(t)\|_{L^{\infty}(\Omega)} \leq C \frac{\|u_0\|_{L^1(\Omega)}}{t^{h_1} e^{\sigma t}}, \quad \forall t > 0,
$$

and universal bounds (i.e. estimates independent on the initial datum $u_0$)

$$
\|u(t)\|_{L^{\infty}(\Omega)} \leq \frac{C}{t^{h_2}}, \quad \forall t > 0,
$$

(see Theorem 2.2 in Sect. 2). Since all the different equations cited before (i.e. the heat equation, the p-Laplacian equation, the porous media equation, the doubly nonlinear equation etc.) satisfy this kind of integral estimates, we can use these results to derive, with an unified proof, a large number of decay estimates.

In particular, we prove here decay estimates for the solutions of Leray–Lions type problems

$$
\begin{cases}
  u_t - \text{div}(a(x, t, u, \nabla u)) = 0 & \text{in } \Omega \times (0, +\infty), \\
  u = 0 & \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x) & \text{on } \Omega,
\end{cases}
$$

(1.4)