Generation of cosine families via Lord Kelvin’s method of images

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Abstract. We show that generation theorems for cosine families related to one-dimensional Laplacians in $C[0, \infty]$ may be obtained by Lord Kelvin’s method of images, linking them with existence of invariant subspaces of the basic cosine family. This allows us to deal with boundary conditions more general than those considered before (Bátkai and Engel in J Differ Equ 207:1–20, 2004; Chill et al. in Functional analysis and evolution equations. The Günter Lumer volume, Birkhäuser, Basel, pp 113–130, 2007; Xiao and Liang in J Funct Anal 254:1467–1486, 2008) and to give explicit formulae for transition kernels of related Brownian motions on $[0, \infty)$. As another application we exhibit an example of a family of equibounded cosine operator functions in $C[0, \infty]$ that converge merely on $C_0(0, \infty]$ while the corresponding semigroups converge on the whole of $C[0, \infty]$.

1. Introduction

Let $C[0, \infty]$ be the space of continuous functions on $[0, \infty)$ with limits at infinity, and $C_0(0, \infty]$ be its subspace of functions vanishing at 0. Also, let $C[-\infty, \infty]$ be the space of continuous functions on $\mathbb{R}$ with limits at plus and minus infinity.

In his classic [9, pp. 340-343], W. Feller constructs the minimal Brownian motion semigroup $(e^{\frac{1}{2} \Delta_m t})_{t \geq 0}$ in $C_0(0, \infty]$ from the unrestricted Brownian motion semigroup $(e^{\frac{1}{2} \Delta t})_{t \geq 0}$ in $C[-\infty, \infty]$ in the following ingenious way, referred to as Lord Kelvin’s method of images (see also e.g. [11,14]). Taking an $f \in C_0(0, \infty]$, he considers its odd extension $\tilde{f}$ to the whole of $\mathbb{R}$ and notes that $(e^{\frac{1}{2} \Delta t})_{t \geq 0}$ leaves the subspace of odd functions invariant. Hence, $e^{\frac{1}{2} \Delta t} \tilde{f}$ equals 0 at $x = 0$, and its restriction to $\mathbb{R}^+$ is a member of $C_0(0, \infty]$ and turns out to be equal to $e^{\frac{1}{2} \Delta_m t} f$. The same argument works for the reflected Brownian motion semigroup $(e^{\frac{1}{2} \Delta_r t})_{t \geq 0}$ in $C[0, \infty]$ except that this time the even extension of $f \in C[0, \infty]$ is needed. In other words, $(e^{\frac{1}{2} \Delta_m t})_{t \geq 0}$ and $(e^{\frac{1}{2} \Delta_r t})_{t \geq 0}$ are similar [8] to $(e^{\frac{1}{2} \Delta t})_{t \geq 0}$ as restricted to the subspaces of odd functions and even functions, respectively. In particular, the domain $D(\Delta_r)$ of the generator of $(e^{\frac{1}{2} \Delta_r t})_{t \geq 0}$ is an isomorphic image of the domain of $(e^{\frac{1}{2} \Delta t})_{t \geq 0}$ as restricted to the subspace of even functions. Therefore, $D(\Delta_r)$ is composed of functions with twice continuously

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differentiable even extensions or, which is the same, twice continuously differentiable functions satisfying the boundary condition \( f'(0) = 0 \); see [5] for details.

A natural question arises whether such a construction can be carried out also in the case of more general boundary conditions, including Robin condition corresponding to the elastic Brownian motion [12,15]. This question seems to have been open for decades; the method of images hinges on constructing an extension of a function \( f \in C[0, \infty) \) to the whole of \( \mathbb{R} \) in such a way that the subspace of extensions of all \( f \in C[0, \infty) \) is invariant for \( (e^{\Delta t})_{t \geq 0} \), but it have been unclear how this extension should be derived from the given boundary condition.

In their recent paper [7], Chill et al. show that one-dimensional Laplacians with Robin boundary conditions generate cosine families, by providing their explicit form in terms of d’Alembert’s formula. Their procedure involves giving explicitly the extension described above; it may be checked directly that the subspace of extensions introduced in [7] is invariant for the canonical cosine family

\[
C(t)f(x) = \frac{1}{2}[f(x + t) + f(x - t)], \quad x, t \in \mathbb{R},
\]

and leads to the desired boundary condition. Because of the Weierstrass formula, linking a cosine family with the related semigroup, this subspace is also invariant for \( (e^{\Delta t})_{t \geq 0} \) and the problem posed above is solved in the case of the elastic Brownian motion.

In our main Theorem 2.2, we extend this result to a quite large class of boundary conditions, including the so-called non-local conditions considered in [3], and most of those in [17]. While the approach presented in [7] seems to hinge on guessing the form of the needed extension, in proving our fundamental Lemma 2.1, we provide an efficient way of deriving this form. Interestingly, due to the simplicity of the basic cosine family and relative complexity of the related semigroup, the method of images turns out to be much simpler for cosine families than it is for semigroups. This should be contrasted with the fact that in general, generation theorems for cosine families are harder to obtain than those for semigroups [16].

In contradistinction to the methods presented in [3,17], the method of images provides closed formulae for the cosine families considered. This intimate knowledge allows us to use the Trotter–Kato convergence theorem to obtain a new cosine family generation theorem (Theorem 2.5) for the one-dimensional Laplacian with more general boundary condition than that considered in [3], and partly in [17]. Our theorem deals with the situation where the measure involved [see (2.1)] needs not be absolutely continuous, but it does not include an equivalent of the so-called periodic case considered in [17].

In Sect. 3, we apply our generation results to give explicit forms of transition probabilities for Brownian motions in \([0, \infty)\) with quite general behavior at the boundary 0; these formulae seem to have been unknown [6]. In Sect. 4, we give growth estimates for cosine families considered and exhibit an example of a sequence of equibounded