Abstract. We examine an infinite system of ordinary differential equations that models a discrete fragmentation process in which mass loss can occur. The problem is treated as an abstract Cauchy problem, posed in an appropriate Banach space. Perturbation techniques from the theory of semigroups of operators are used to establish the existence and uniqueness of physically meaningful solutions under minimal restrictions on the fragmentation rates. In one particular case, an explicit formula for the associated semigroup is obtained and this enables additional properties, such as compactness of the resolvent and analyticity of the semigroup, to be deduced. Another explicit solution of this particular fragmentation problem, in which mass is apparently created from a zero-mass initial state, is also investigated, and the theory of Sobolev towers is used to prove that the solution actually emanates from an initial infinite cluster of unit mass.

1. Introduction

Coagulation and fragmentation processes occur in many areas of pure and applied sciences such as particle and aerosol physics. In a system where only fragmentation can take place, clusters of particles break up into a number of smaller clusters. As coagulation does not occur in such systems, the process governing the evolution of the cluster-size distribution is irreversible. If we assume that each cluster of size \( n \in \mathbb{N} \) (an \( n \)-mer) is composed of \( n \) identical fundamental units (monomers), then the mass of each cluster is simply a positive integer multiple of the mass of the monomer. By appropriate scaling, each monomer can be assumed to have unit mass. This leads to a discrete model of the fragmentation process, in which the evolution of clusters is described by

\[
\frac{d}{dt} u_k(t) = -a_k u_k(t) + \sum_{j=k+1}^{\infty} b_{k,j} a_j u_j(t), \quad t > 0, \quad k = 1, 2, 3, \ldots \quad (1.1)
\]

In (1.1), \( u_k(t) \) represents the density of \( k \)-mers at time \( t \), \( a_k \) is the average breakup rate of a \( k \)-mer and \( b_{k,j} \) is the average number of \( k \)-mers produced upon the breakup of a \( j \)-mer. In most investigations into discrete models of fragmentation, it is usually

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assumed that the total mass, $M(t)$, in the system is a conserved quantity. For the model described by (1.1), this requires

$$a_1 = 0 \quad \text{and} \quad \sum_{k=1}^{j-1} kb_{k,j} = j, \quad j = 2, 3, \ldots.$$  \hfill (1.2)

A simple calculation then shows that, formally,

$$\frac{d}{dt} M(t) = \frac{d}{dt} \sum_{k=1}^{\infty} ku_k(t) = 0. \hfill (1.3)$$

The customary way of analysing the infinite systems of differential equations, such as (1.1), that arise as discrete models of fragmentation is to consider, initially, finite-dimensional truncations. Standard methods from the theory of ordinary differential equations then lead to the existence of a sequence of solutions to these truncated equations, and compactness arguments establish that there is a subsequence that converges to a solution $u(t) = (u_1(t), u_2(t), u_3(t), \ldots)$ of an integral version of the infinite system. This strategy has been used, for example, by Laurençot in [10] for the system (1.1), and by Ball and Carr [1] and da Costa [7] for the case of discrete binary fragmentation.

In contrast to the truncation-limit approach, a more operator-based strategy was adopted by three of the current authors in [12] to establish the existence and uniqueness of solutions to (1.1) for suitably restricted initial data. This approach avoided truncations, relying instead on perturbation results for semigroups of operators that had proved beneficial in a number of previous investigations into continuous fragmentation models, including cases where a loss of overall mass, due to phenomena such as oxidation, is incorporated in the model; see [4, Chapters 8–9]. As pointed out by Cai, Edwards and Han [6], mass loss can also be a feature of discrete processes modelled by (1.1). In such cases, (1.2) and (1.3) are replaced, respectively, by

$$\sum_{k=1}^{j-1} kb_{k,j} = (1 - \lambda_j) j, \quad j = 2, 3, \ldots,$$  \hfill (1.4)

and

$$\frac{d}{dt} M(t) = -a_1 u_1(t) - \sum_{k=2}^{\infty} k\lambda_k a_k u_k(t), \hfill (1.5)$$

where $\lambda_j \in [0, 1]$, the mass-loss fraction, represents the loss of clusters of mass $j$. It should be noted that we no longer require $a_1$ to be zero. Also, when $a_1 = 0$ and $\lambda_j = 0$ for all $j \geq 2$, then we recover the mass-conserving case.

To motivate mass-loss models of this type, Cai et al. [6,8] discuss the specific case of random bond annihilation, where a “bond” is interpreted as a unit line segment joining two integer values on the real line. A $k$-chain is a set of $k + 1$ consecutive