Uniqueness of diffusion operators and capacity estimates

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Abstract. Let \( \Omega \) be a connected open subset of \( \mathbb{R}^d \). We analyse \( L_1 \)-uniqueness of real second-order partial differential operators \( H = -\sum_{k,l=1}^d \partial_k c_{kl} \partial_l \) and \( K = H + \sum_{k=1}^d c_k \partial_k + c_0 \) on \( \Omega \) where \( c_{kl} = c_{lk} \in W^{1,\infty}_{\text{loc}}(\Omega) \), \( c_k \in L_{\infty,\text{loc}}(\Omega) \), \( c_0 \in L_{2,\text{loc}}(\Omega) \) and \( C(x) = (c_{kl}(x)) > 0 \) for all \( x \in \Omega \). Boundedness properties of the coefficients are expressed indirectly in terms of the balls \( B_\eta \) existent in \([16]\) for bounded coefficients and their Lebesgue measure \(|B(r)|\). First, we establish that if the balls \( B(r) \) are bounded, the Täcklind condition \( \int_\mathbb{R}^d dr r (\log |B(r)|)^{-1} = \infty \) is satisfied for all large \( R \) and \( H \) is Markov unique then \( H \) is \( L_1 \)-unique. If, in addition, \( C(x) \geq \kappa (c^T \otimes c)(x) \) for some \( \kappa > 0 \) and almost all \( x \in \Omega \), \( \text{div} \, c \in L_{\infty,\text{loc}}(\Omega) \) is upper semi-bounded and \( c_0 \) is lower semi-bounded, then \( K \) is also \( L_1 \)-unique. Secondly, if the \( c_{kl} \) extend continuously to functions which are locally bounded on \( \partial \Omega \) and if the balls \( B(r) \) are bounded, we characterize Markov uniqueness of \( H \) in terms of local capacity estimates and boundary capacity estimates. For example, \( H \) is Markov unique if and only if for each bounded subset \( A \) of \( \Omega \) there exist \( \eta_n \in C_{\infty}^1(\Omega) \) satisfying \( \lim_{n \to \infty} \| A \, \Gamma(\eta_n) \|_1 = 0 \), where \( \Gamma(\eta_n) = \sum_{k,l=1}^d c_{kl} (\partial_k \eta_n)(\partial_l \eta_n) \), and \( \lim_{n \to \infty} \| A (\eta_n - \eta_0) \, \varphi \|_2 = 0 \) for each \( \varphi \in L_2(\Omega) \) or if and only if \( \text{cap}(\partial \Omega) = 0 \).

1. Introduction

Let \( \Omega \) be a connected open subset of \( \mathbb{R}^d \) and define the second-order divergence-form operator \( H \) on the domain \( D(H) = C_\infty^1(\Omega) \) by

\[
H = -\sum_{k,l=1}^d \partial_k c_{kl} \partial_l
\]

where the \( c_{kl} = c_{lk} \) are real-valued functions in \( W^{1,\infty}_{\text{loc}}(\Omega) \), and the matrix \( C = (c_{kl}) \) is strictly elliptic, that is, \( C(x) > 0 \) for all \( x \in \Omega \). It is possible that the coefficients can have degeneracies as \( x \to \partial \Omega \), the boundary of \( \Omega \) or as \( x \to \infty \).

The operator \( H \) is defined to be \( L_1 \)-unique if it has a unique \( L_1 \)-closed extension which generates a strongly continuous semigroup on \( L_1(\Omega) \). Alternatively, it is defined to be Markov unique if it has a unique \( L_2 \)-closed extension which generates a submarkovian semigroup on the spaces \( L_p(\Omega) \). Markov uniqueness is a direct consequence of \( L_1 \)-uniqueness since distinct submarkovian extensions give distinct \( L_1 \)-extensions. But the converse implication is not valid in general. The converse was established in [16] for bounded coefficients \( c_{kl} \), and the proof was extended in [17].
to allow a growth of the coefficients at infinity. The converse can, however, fail if the coefficients grow too rapidly (see [17] Section 4.1). The principal aim of the current paper is to establish the equivalence of Markov uniqueness and $L_1$-uniqueness of $H$ from properties of the Riemannian geometry defined by the metric $C^{-1}$ which give, implicitly, optimal growth bounds on the coefficients.

Our arguments extend to non-symmetric operators

$$K = H + \sum_{k=1}^{d} c_k \frac{\partial}{\partial x_k} + c_0$$

(2)

with the real-valued lower-order coefficients satisfying the following three conditions:

1. $c_0 \in L_{2,\text{loc}}(\Omega)$ is lower semi-bounded,
2. $c_k \in L_{\infty,\text{loc}}(\Omega)$ for each $k = 1, \ldots, d$, $\text{div} \, c \in L_{\infty,\text{loc}}(\Omega)$ and $\text{div} \, c$ is upper semi-bounded,
3. there is a $\kappa > 0$ such that $C(x) \geq \kappa (c^T \otimes c)(x)$ for almost all $x \in \Omega$.

(3)

In the second condition, $c = (c_1, \ldots, c_d)$ and $\text{div} \, c = \sum_{k=1}^{d} \partial_k c_k$ with the partial derivatives understood in the distributional sense. The third condition in (3) is understood in the sense of matrix ordering, that is, $(c_{kl}(x)) \geq \kappa (c_k(x)c_l(x))$ for almost all $x \in \Omega$. These conditions together with the general theory of accretive sectorial forms are sufficient to ensure that $K$ has an extension which generates a strongly continuous semigroup on $L_1(\Omega)$ (see Sect. 2). As in the symmetric case, $K$ is defined to be $L_1$-unique if it has a unique such extension.

The Riemannian distance $d(\cdot ; \cdot)$ corresponding to the metric $C^{-1}$ can be defined in various equivalent ways but in particular by

$$d(x ; y) = \sup\{\psi(x) - \psi(y) : \psi \in W^{1,\infty}_{\text{loc}}(\Omega), \Gamma(\psi) \leq 1\}$$

(4)

for all $x, y \in \Omega$ where $\Gamma$, the carré du champ of $H$, denotes the positive map

$$\varphi \in W^{1,2}_{\text{loc}}(\Omega) \mapsto \Gamma(\varphi) = \sum_{k,l=1}^{d} c_{kl}(\partial_k \varphi)(\partial_l \varphi) \in L_{1,\text{loc}}(\Omega).$$

(5)

Since $\Omega$ is connected and $C > 0$, it follows that $d(x ; y)$ is finite for all $x, y \in \Omega$ but one can have $d(x ; y) \to \infty$ as $x, y$ tends to the boundary $\partial \Omega$. Throughout the sequel, we choose coordinates such that $0 \in \Omega$ and denote the Riemannian distance to the origin by $\rho$. Thus, $\rho(x) = d(x ; 0)$ for all $x \in \Omega$. The Riemannian ball of radius $r > 0$ centred at $0$ is then defined by $B(r) = \{x \in \Omega : \rho(x) < r\}$, and its volume (Lebesgue measure) is denoted by $|B(r)|$.

There are two properties of the balls $B(r)$ which are important in our analysis. First, the balls $B(r)$ must be bounded for all $r > 0$. It follows straightforwardly that this is equivalent to the condition that $\rho(x) \to \infty$ as $x \to \infty$, that is, as $x$ leaves any