A mixed problem of linear elastodynamics

KAZUAKI TAIRA

Abstract. This paper is devoted to a semigroup approach to an initial-boundary value problem of linear elastodynamics in the case where the boundary condition is a regularization of the genuine mixed displacement-traction boundary condition. More precisely, it is a smooth linear combination of displacement and traction boundary conditions, but is not equal to the pure traction boundary condition. Some previous results with mixed displacement-traction boundary condition are due to Inoue and Ito. The crucial point in our semigroup approach is to generalize the classical variational approach to the degenerate case, by using the theory of fractional powers of analytic semigroups.

1. Introduction

This paper is devoted to a semigroup approach to an initial-boundary value problem of linear elastodynamics in the case where the boundary condition is a regularization of the genuine mixed displacement-traction boundary condition. More precisely, it is a smooth linear combination of displacement and traction boundary conditions, but is not equal to the pure traction boundary condition. Some previous results with mixed displacement-traction boundary condition are due to Inoue [14, 15] and Ito [16]. Many problems in partial differential equations can be formulated in terms of abstract operators acting between suitable spaces of distributions, and these operators are then analyzed by the methods of semigroup theory. The crucial point in our semigroup approach is to generalize the classical variational approach to the degenerate case, by using the theory of fractional powers of analytic semigroups. The virtue of this approach is that a given problem is stripped of extraneous data, so that the analytic core of the problem is revealed [7, 24].

Let \( \Omega \) be an open, connected subset of Euclidean space \( \mathbb{R}^n, n \geq 2 \), with smooth boundary \( \partial \Omega \). We think of the closure \( \overline{\Omega} = \Omega \cup \partial \Omega \) of \( \Omega \) as representing the volume occupied by an undeformed body; so the set \( B = \overline{\Omega} \) is called the reference configuration (cf. [4, 17, 23]). In this paper, we study the following initial-boundary value problem of linear elastodynamics (see [17, Chapter 6, Section 6.3]): For given vector functions

Mathematics Subject Classification (2010): 74B15, 47D03, 35J57

Keywords: Initial-boundary value problem, Linear elastodynamics, Displacement-traction boundary condition, Contraction group.

Dedicated to Professor Mitsuru Ikawa on the occasion of his 70th birthday.
\[ f(x) = (f_i(x)), \quad u_0(x) = (u_{0,i}(x)) \text{ and } u_1(x) = (u_{1,i}(x)) \] defined in \( \Omega \), find a vector function \( u(x, t) = (u_i(x, t)) \) in \( \Omega \times (0, \infty) \) such that

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \text{div}(a(x) \cdot \nabla u) &= f \
|u|_{t=0} &= u_0 \
|\frac{\partial u}{\partial t}|_{t=0} &= u_1 \
\alpha(x)(a(x) \cdot \nabla u \cdot n) + (1 - \alpha(x))u &= 0
\end{align*}
\]

(1.1)

Here:

1. \( a(x) = (a_{ij\ell m}(x)) \) is a smooth elasticity tensor.
2. \( \alpha(x) \) is a smooth real-valued function on \( \partial \Omega \) such that \( 0 \leq \alpha(x) \leq 1 \) on \( \partial \Omega \).
3. \( n = (n_i) \) is the outward unit normal to \( \partial \Omega \).

It is worth pointing out that, componentwise, the initial-boundary value problem (1.1) can be written in the form

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \sum_{\ell,m=1}^{n} a_{ij\ell m}(x) \frac{\partial u_\ell}{\partial x_m} \right) &= f_i(x) \
|u|_{t=0} &= u_{0,i}(x) \
|\frac{\partial u_i}{\partial t}|_{t=0} &= u_{1,i}(x) \
\alpha(x) \sum_{j=1}^{n} \left( \sum_{\ell,m=1}^{n} a_{ij\ell m}(x) \frac{\partial u_\ell}{\partial x_m} \right) n_j(x) + (1 - \alpha(x))u_i(x) &= 0
\end{align*}
\]

on \( \partial \Omega \times (0, \infty) \).

It should be noticed that our boundary condition

\[ B_\alpha u = \alpha(x)(a(x) \cdot \nabla u \cdot n) + (1 - \alpha(x))u \]

is a smooth linear combination of displacement and traction boundary conditions. It is easy to see that \( B_\alpha \) is non-degenerate (or coercive) if and only if either \( \alpha(x) > 0 \) on \( \partial \Omega \) (the Robin case) or \( \alpha(x) \equiv 0 \) on \( \partial \Omega \) (the Dirichlet case). Marsden–Hughes [17] studied the non-degenerate case. More precisely, they assume that the boundary \( \partial \Omega \) is the disjoint union of the two closed subsets \( \Gamma_0 = \{ x \in \partial \Omega : \alpha(x) = 0 \} \) and \( \partial \Omega \setminus \Gamma_0 = \{ x \in \partial \Omega : \alpha(x) > 0 \} \).

However, our boundary condition \( B_\alpha \) is degenerate from an analytical point of view. This is due to the fact that the so-called Shapiro–Lopatinskii complementary condition is violated at the points \( x \in \partial \Omega \) where \( \alpha(x) = 0 \) (cf. [11]). For example, in the case where \( n = 3 \), \( \alpha(x) \) may be a function such that, in terms of local coordinates \( (x_1, x_2) \) of \( \partial \Omega \),

\[ \alpha(x) = e^{-1/x_1^2} \sin^2 \frac{1}{x_1} e^{-1/x_2^2} \sin^2 \frac{1}{x_2}. \]

Therefore, the crucial point in our semigroup approach is how to generalize the classical variational approach to the degenerate case (see Sect. 2.1).

We give two simple but important examples of the initial-boundary value problem (1.1):