On non-autonomous evolutionary problems

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Abstract. The paper extends well-posedness results of a previously explored class of time-shift invariant evolutionary problems to the case of non-autonomous media. The Hilbert space setting developed for the time-shift invariant case can be utilized to obtain an elementary approach to non-autonomous equations. The results cover a large class of evolutionary equations, where well-known strategies like evolution families may be difficult to use or fail to work. We exemplify the approach with an application to a Kelvin–Voigt-type model for visco-elastic solids.

0. Introduction

In a number of studies, it has been demonstrated that systems of the form

\[(\partial_0 M + A) u = F, \]

(0.1)

where \(M\) is a continuous, linear mapping, and the densely defined, closed linear operator \(A\) is such that \(A\) and \(A^*\) are maximal \(\omega\)-accretive for some suitable \(\omega \in \mathbb{R}\), cover numerous models from mathematical physics. Indeed, \(A\) skew-selfadjoint\(^1\) is a standard situation, which for simplicity we shall assume throughout. The well-posedness of (0.1) hinges on a positive definiteness assumption imposed on \(\partial_0 M\) in a suitable space-time Hilbert space setting. Under this assumption, the solution theory is comparatively elementary since \((\partial_0 M + A)\) together with its adjoint are positive definite yielding that \((\partial_0 M + A)\) has dense range and a continuous inverse.

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\(^1\) Two densely defined linear operators \(A, B\) are skew-adjoint (to each other) if \(A = -B^*\). If \(A = B\), we call \(A\) skew-selfadjoint (rather than self-skew-adjoint). The proper implications

A selfadjoint \(\implies\) A symmetric \(\implies\) A Hermitean

are paralleled by

A skew-selfadjoint \(\implies\) A skew-symmetric \(\implies\) A skew-Hermitean.

Frequently, “skew-adjoint” is used to mean “skew-selfadjoint.” We shall, however, not follow this custom for the obvious reason.
In applications of this setting, the operator $A$ has a rather simple structure, whereas the complexity of the physical system is encoded in the “material law” operator $\mathcal{M}$. A simple but important case is given by

$$\mathcal{M} = M_0 + \partial_0^{-1} M_1,$$

where $M_0, M_1$ are time-independent continuous linear operators. Here, we have anticipated that in the Hilbert space setting to be constructed $\partial_0^{-1}$ (time integration) has a proper meaning. The positive definiteness assumption requires $M_0$ to be nonnegative and selfadjoint and

$$\varrho M_0 + \Re M_1 \geq c_0 \quad (0.2)$$

for some $c_0 \in ]0, \infty[$ and all sufficiently large $\varrho \in ]0, \infty[$. Since we do not assume that $M_0$ is always strictly positive, (0.2) may imply constraints on $M_1$. If $M_0$ is positive definite, it may seem natural, following the proven idea of first finding a fundamental solution (given by an associated semi-group), and then to obtain general solutions as convolutions with the data (Duhamel’s principle, variation of constants formula) and so proving well-posedness. This is the classical method of choice in a Banach space setting, see e.g., [1,5,6,9,18] as general references. In comparison, our approach is (currently) limited to a Hilbert space setting; however, apart from being conceptually more elementary, it allows to incorporate delay and convolution integral terms by a simple perturbation argument and, if $M_0$ has a non-trivial kernel, the system becomes a differential–algebraic systems, which to the above approach makes no difference, but cannot be conveniently analyzed within the framework of semi-group theory.

The purpose of this paper is to extend well-posedness results previously obtained for time-shift invariant material operators $\mathcal{M}$ to cases, where $\mathcal{M}$ is not time-shift invariant. This is the so-called time-dependent or non-autonomous case. The above-mentioned limitations of the semi-group approach carry over to the application of classical strategies based on evolution families introduced by Kato [8], which correspond to abstract Green’s functions for non-autonomous differential equations, for a survey see e.g. [9,17]. On the other hand, the evolution family approach allows for non-autonomous evolution equations involving a time-dependent operator family with varying domain, which is currently beyond the scope of the framework presented here. As a particular instance, we refer to [16], where such non-autonomous equations of hyperbolic type are treated and the time-asymptotic behavior of solutions is studied. In contrast, the approach we shall develop here bypasses the relative sophistication of the classical approach based on evolution families and, as a consequence, allows to extend the solution theory to include, for example, differential–algebraic systems and memory effects in a simple unified setting of broad applicability.

To keep the presentation self-contained, we construct the Hilbert space setting in sufficient detail and formulate our results so that the autonomous case re-appears as a special case of the general non-autonomous situation. The key will be defining the non-autonomous problem in a space-time setting as a sum of two (unbounded) operators.