Null controllability of $n$-coupled degenerate parabolic systems with $m$-controls

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Abstract. In this paper we will analyze the null controllability properties of a linear coupled degenerate parabolic system of $n$ equations when $m$ distributed controls are exerted on the system. First we start with the case when the coupling matrix $A$ is cascade, and then when $A$ is a full matrix, we will prove that the Kalman rank condition on the coupling and the control matrices $A$ and $B$ characterizes the controllability properties of the system.

1. Introduction

In this work, we study the null controllability properties of degenerate parabolic systems under several control forces on some nonempty open set $\omega \subset (0, 1)$. All along this work, we will denote $Q := (0, T) \times (0, 1)$ and $\Sigma := (0, T) \times \{0, 1\}$, for $T > 0$. Consider the following degenerate parabolic linear system of $n$-coupled parabolic degenerate equations, with $m$ control forces:

$$
\begin{align*}
\partial_t y_1 - d_1 L y_1 + \sum_{j=1}^{n} a_{1j}(t, x) y_j &= b_1 \mathbb{1}_\omega v_1(t, x) \quad \text{in } Q, \\
\partial_t y_2 - d_2 L y_2 + \sum_{j=1}^{n} a_{2j}(t, x) y_j &= b_2 \mathbb{1}_\omega v_2(t, x) \quad \text{in } Q, \\
&\vdots \\
\partial_t y_m - d_m L y_m + \sum_{j=1}^{n} a_{mj}(t, x) y_j &= b_m \mathbb{1}_\omega v_m(t, x) \quad \text{in } Q, \\
\partial_t y_{m+1} - d_{m+1} L y_{m+1} + \sum_{j=1}^{n} a_{m+1j}(t, x) y_j &= 0 \quad \text{in } Q, \\
&\vdots \\
\partial_t y_n - d_n L y_n + \sum_{j=1}^{n} a_{nj}(t, x) y_j &= 0 \quad \text{in } Q,
\end{align*}
$$

(1.1)

where $L$ is a strongly elliptic operator, and $\mathbb{1}_\omega$ denotes the characteristic function of $\omega$. The initial data are given by:

$$
\begin{align*}
y_1(0, x) &= y_1(x) \quad 1 \leq i \leq n \\
y_i(0, x) &= y_0(x) \quad 1 \leq i \leq n 
\end{align*}
$$

in $(0, 1)$.

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The differential operator \( L \) is defined by
\[
Ly = (a(x)y_x)_x,
\]
where \( a(x) \) is a diffusion coefficient which can degenerate at 0, (i.e., \( a(0) = 0 \)). The weak (WD) and strong (SD) degenerate cases depend on the diffusion coefficient \( a \) and are specified in Sect. 2, the couplage terms \( a_{ij} = a_{ij}(t, x) \) \( L^\infty(Q) \), \( y_{0,i} \in L^2(0, 1) \), \( 1 \leq i \leq n \), \( d_i > 0 \), \( 1 \leq i \leq n \), and \( \mathbb{1}_\omega \) is the characteristic function of the nonempty control region \( \omega \).

Equivalently, the previous system can be written as
\[
\begin{aligned}
\begin{cases}
\partial_t Y - L Y + A(t)Y &= B v \mathbb{1}_\omega \quad \text{in } Q, \\
CY &= 0 \quad \text{on } \Sigma, \\
Y(0, x) &= Y_0(x) \quad \text{in } (0, 1),
\end{cases}
\end{aligned}
\tag{1.2}
\]
where the operator \( L \) is given by
\[
L = DL
\]
\[
D = \text{diag}(d_1, \ldots, d_n)
\]
The boundary operator \( C \) is either the trace operator in the weak degenerate case or the associated Neumann boundary condition in the strongly degenerate case.

The matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \) has its entries in \( L^\infty(Q) \) and \( B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \) and \( v = (v_1, \ldots, v_m)^* \in L^2(Q, \mathbb{R}^m) \) is the vector control.

The adjoint system associated with the system (1.2) is the following
\[
\begin{aligned}
\begin{cases}
-\partial_t \varphi - L \varphi + A^*(t)\varphi &= 0 \quad \text{in } Q, \\
C \varphi &= 0 \quad \text{on } \Sigma, \\
\varphi(T) &= \varphi_T \quad \text{in } (0, 1).
\end{cases}
\end{aligned}
\tag{1.3}
\]
In this paper we study the system (1.2) specifically in the following two cases:

- the diffusion matrix \( D = (d_1, \ldots, d_n) \), \( d_i > 0 \) and the coupling matrix \( A \) is cascade with conditions on subdiagonal terms of \( A \)
- the diffusion matrix \( D = dI_n \) and the coupling matrix \( A \) is a full matrix with the Kalman rank condition \( \text{rank}[A | B] = n \).

It is well known that the null controllability of system (1.2) is equivalent to the existence of a constant \( C > 0 \) such that every solution \( \varphi = (\varphi_1, \ldots, \varphi_n)^* \) to adjoint system (1.3) satisfies
\[
\| \varphi(\cdot, 0) \|_{L^2(0, 1)^n}^2 \leq C \iint_{\omega \times (0, T)} |B^*\varphi(x, t)|^2 \, dx \, dt.
\tag{1.4}
\]
To get the observability inequality (1.4) we reason as follows: Firstly we will transform our system (1.2) to a cascade one. Secondly we state a new Carleman estimates for the solutions to the adjoint of the cascade system see Theorem 3.8 and from this result one can finally deduces the desired inequality.