On the characterization of complex Shimura varieties

Yakov Varshavsky

Abstract. In this paper we recall basic properties of complex Shimura varieties and show that they actually characterize them. This characterization immediately implies the explicit form of Kazhdan’s theorem on the conjugation of Shimura varieties. It also implies the existence of unique equivariant models over the reflex field of Shimura varieties corresponding to adjoint groups and the existence of a $p$-adic uniformization of certain unitary Shimura varieties. In the appendix we give a modern formulation and a proof of Weil’s descent theorem.

Mathematics Subject Classification (2000). 11418, 14435.

Key words. Shimura varieties, Weil descent.

1. Introduction

Shimura varieties are natural generalizations of classical modular curves. As complex manifolds, Shimura varieties are just finite disjoint unions of quotients of Hermitian symmetric domains by arithmetic (congruence) subgroups. By the theorem of Baily-Borel, they have canonical structures of complex quasi-projective varieties.

In some cases Shimura varieties have a modular interpretation as spaces parametrizing families of polarized abelian varieties with some additional structure, or, more generally, of abelian motives. This interpretation have proved to be very useful for establishing various properties of Shimura varieties, such as for showing that they have canonical structures (models) of quasi-projective varieties defined over certain number fields, called the reflex fields.

Another (analytic) approach to the study of Shimura varieties was introduced by Kazhdan in [Ka1], [Ka2]. In those papers he showed that every conjugate of a Shimura variety is again a Shimura variety (a much better proof of this fact was given by Nori and Raghunathan in [NR]). This result combined with some modular and group-theoretic methods allows one to prove the existence of the canonical models for all Shimura varieties (see [Brv], [Mi1] and [Mi3] or [Mo, Thm. 2.18]).
The strategy of Kazhdan’s proof (and that of Nori and Raghunathan) was to show that Shimura varieties are so rigid that a conjugate of a Shimura variety cannot be anything but a certain, possibly different Shimura variety.

One of the main goals of this paper is to make this strategy explicit, that is to write down a list of properties which characterize Shimura varieties uniquely. For this we consider not just a Shimura variety but a triple consisting of a Shimura variety $X$, a standard $G$-torsor $P$ with a flat connection $H$ over it, and an equivariant map $\rho$ from $P$ to a certain generalized Grassmannian $Gr$. More precisely, we consider an adelic family of the above triples, and our Main Theorem (see Section 4) asserts that our family is the unique family, satisfying certain group-theoretic properties.

The strategy of the proof is as follows: let $(X; (P, H); \rho : P \to Gr)$ be a triple as above; we want to show that it is associated to a certain Shimura variety. Replacing $X$ and $P$ by their connected components, we may assume that they are connected. Let $M$ be the universal cover of the complex manifold $X^{an}$, and let $\Gamma$ be its fundamental group. Then $X^{an}$ is isomorphic to $\Gamma \backslash M$, and our goal is to show that certain properties of the triple imply that $M$ is a Hermitian symmetric domain, and $\Gamma \subset \text{Aut}(M)$ is an arithmetic (congruence) subgroup.

Using the flat connection $H$ on $P$, we conclude that the principal bundle $P^{an}$ on $X^{an}$ is isomorphic to $\Gamma \backslash [M \times G^{an}]$. Then the composition of the embedding $M \cong M \times \{1\} \hookrightarrow P^{an}$ and $\rho$ gives us a $\Gamma$-equivariant morphism $\rho_0 : M \to Gr^{an}$. Since our triple is a part of an adelic family, we conclude from our properties that $\rho_0$ is a local isomorphism and that it is equivariant with respect to a much larger group $\tilde{\Gamma} \supset \Gamma$. Using the fact that $Gr^{an}$ is a compact Hermitian symmetric space, our properties guarantee that $M$ is the Hermitian symmetric domain, dual to $Gr^{an}$. Since $\Gamma \backslash M$ is a quasi-projective variety, we get that $\Gamma$ is a lattice in the semisimple Lie group $\text{Aut}(M)$, and our properties imply that it is irreducible and has an infinite index in its commensurator. Hence, by Margulis’ theorem, $\Gamma$ is arithmetic, as claimed.

Since the properties, characterizing the triple, are stable under the conjugation, our Main Theorem immediately implies the explicit form of Kazhdan’s theorem, which is a weak form of Langlands’ conjecture. Also our Main Theorem implies the existence of the $p$-adic uniformization for some unitary Shimura varieties, proved in [RZ] and [Va1], [Va2], which is essentially just a non-archimedean analog of Kazhdan’s theorem. Therefore our result can serve as an explanation why both Kazhdan’s theorem and the theorem on the $p$-adic uniformization hold.

Using the fact that Shimura varieties corresponding to adjoint groups have no non-trivial equivariant automorphisms, we then show that such Shimura varieties have unique equivariant structures over their reflex fields (which coincide by uniqueness with the canonical models in the sense of Deligne). For this we show that the descent datum, obtained from Kazhdan’s theorem is effective. Moreover, the reflex field is the minimal subfield of $\mathbb{C}$ over which such an equivariant model