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Reconstruction of groups

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Introduction

The classical results in this area are the Pontryagin duality theorem for abelian locally compact groups and the Tannaka–Krein duality for arbitrary compact groups. Note that the Tannaka–Krein duality reflects the fact that any compact group is the set of real points of an affine pro-algebraic group ('pro-' stands for projective limit). This observation had lead to an extension of Tannaka–Krein duality to pro-algebraic groups ([D], [DM], [R1], [S]) and Lie algebras ([H-C], [R1]).

Continuous finite-dimensional representations of compact groups are rigid in the following sense: every subcategory of the category $f.\operatorname{Rep}(G)$ of finite-dimensional representations of a compact group G, which separates elements of G and is closed under the tensor product and complex conjugation, is naturally equivalent to the category $f.\operatorname{Rep}(G)$ itself. This is a consequence of a similar rigidity property of regular representations of affine pro-algebraic groups. It follows from the results of this paper that infinite-dimensional representations of an arbitrary locally compact or Lie group do not possess the rigidity property. Besides, the category of all, say unitary, representations of a generic locally compact or Lie group is too large, so that a reasonable question in the case of noncompact groups and infinite-dimensional representations is: *what are conditions on a monoidal subcategory of the category of continuous representations of a locally compact group sufficient to reconstruct the group?*

Quite a few attempts were made to find an answer. The first general (i.e., valid for general locally compact groups) result on the topic is due to Tatsuuma [Tat]. He proves (under some additional restrictions of a technical nature) that if a monoidal subcategory $\mathfrak{C}$ of unitary representations of a locally compact group G contains the regular representation of G in the Hilbert space $L^2(G, d_l)$ of functions on G with

integrable square with respect to a left invariant measure $d_l g$ on G, then the group G can be reconstructed from the forgetful functor $\mathfrak{C} \longrightarrow \mathfrak{Hilb}$.

One of the results of this paper is the following

Theorem. *Let G be a locally compact group, and let $\mathfrak{C}$ be a monoidal subcategory of the category of continuous uniformly bounded representations of G in reflexive locally convex vector spaces with real structure. Suppose that the category $\mathfrak{C}$ contains with any representation its dual, is closed under complex conjugation, and separates elements of G (i.e., the intersection of kernels of representations of $\mathfrak{C}$ is the trivial subgroup). If among nonzero matrix elements of representations of $\mathfrak{C}$ there is one which belongs to $L^p(G, d_l)$ for some $p > 0$, then the group G is reconstructed from $\mathfrak{C}$ via a certain canonical procedure.*

Pontryagin duality for abelian locally compact groups, Tannaka–Krein duality for compact groups and the Tatsuuma reconstruction theorem [Tat] are simple consequences of this fact.

The formulation and the first proof of the theorem were found a long time ago [R3]. Here it appears as a corollary of a considerably more general assertions which are applicable in a more general setting. This article, however, does not contain any applications beyond the classical problem of the reconstruction of locally compact groups. Some of the applications are based on facts discovered years after the first proof of the theorem was written.

The paper is organized as follows.

In Section 1 we recall the notions of monoidal categories and monoidal functors and introduce $\star$-categories which are monoidal $\mathbb{C}$-categories with an anti-involution.

In Section 2 we explain the duality formalism Groups $\longleftrightarrow$ {Monoidal functors with a fixed target} and formulate the reconstruction problem for a topological group.

In Section 3 we study some simple properties of the algebra of matrix elements of a $\star$-subcategory $\mathfrak{C}$ of the category of representations. We show that the group $G(F_{\mathfrak{C}})$ dual to $\mathfrak{C}$ (more exactly, to the corresponding forgetful functor $F_{\mathfrak{C}}$) acts on the algebra of the matrix elements.

Section 4, 'Conditions of continuity', contains the main technical assertion which is used later to show that the dual group $G(F_{\mathfrak{C}})$ acts on the matrix algebra $A(F_{\mathfrak{C}})$ by continuous operators. The latter fact is a crucial step towards the reconstruction.

In Section 5 the reconstruction theorems are proved. The classical duality theorems by Pontryagin, Tannaka–Krein and Tatsuuma are recovered as simple corollaries. Another consequence of the reconstruction theorems and some results on matrix elements of irreducible representations of semisimple Lie groups [BW] is the following fact which might be regarded as a one-sided rigidity property of semisimple groups.