Quivers with potentials and their representations I: Mutations

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Abstract. We study quivers with relations given by noncommutative analogs of Jacobian ideals in the complete path algebra. This framework allows us to give a representation-theoretic interpretation of quiver mutations at arbitrary vertices. This gives a far-reaching generalization of Bernstein–Gelfand–Ponomarev reflection functors. The motivations for this work come from several sources: superpotentials in physics, Calabi–Yau algebras, cluster algebras.

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Contents

1. Introduction 60
2. Quivers and path algebras 64
3. Potentials and their Jacobian ideals 67
4. Quivers with potentials 70
5. Mutations of quivers with potentials 79
6. Some mutation invariants 85
7. Nondegenerate QPs 88
8. Rigid QPs 91
9. Relation to cluster-tilted algebras 95
10. Decorated representations and their mutations 98
11. Some three-vertex examples 110
12. Some open problems 115
Acknowledgments 118
References 118
1. Introduction

The main objects of study in this paper are quivers with potentials (QPs for short). Roughly speaking, a QP is a quiver $Q$ together with an element $S$ of the path algebra of $Q$ such that $S$ is a linear combination of cyclic paths. We associate to $S$ the two-sided ideal $J(S)$ in the path algebra generated by the (noncommutative) partial derivatives of $S$ with respect to the arrows of $Q$. We refer to $J(S)$ as the Jacobian ideal, and to the quotient of the path algebra modulo $J(S)$ as the Jacobian algebra. They appeared in physicists’ work on superpotentials in the context of the Seiberg duality in mirror symmetry (see, e.g., [17, 2, 6]). Since in some of their work the superpotentials are required to satisfy some form of Serre duality, we prefer not to use this terminology, and just refer to $S$ as a potential; another reason for this is that we are working with the completed path algebra, so our potentials are possibly infinite linear combinations of cyclic paths. The Jacobian algebras also play an important role in the recent work on Calabi–Yau algebras [4, 25, 26, 27].

In this paper we introduce and study mutations for QPs and their (decorated) representations. In the context of Calabi–Yau algebras, the mutations were discussed in [26] but our approach is much more elementary and down-to-earth. Namely, we develop the setup that directly extends to QPs the Bernstein–Gelfand–Ponomarev reflection functors [3] and their “decorated” version [28].

The original motivation for our study comes from the theory of cluster algebras introduced and studied in a series of papers [18, 19, 1, 20]. In this paper, we deal only with the underlying combinatorics of this theory embodied in skew-symmetrizable integer matrices and their mutations. Furthermore, we restrict our attention to skew-symmetric integer matrices. Such matrices can be encoded by quivers without loops and oriented 2-cycles. Namely, a skew-symmetric integer $n \times n$ matrix $B = (b_{i,j})$ corresponds to a quiver $Q(B)$ with vertices $1, \ldots, n$, and $b_{i,j}$ arrows from $j$ to $i$ whenever $b_{i,j} > 0$. For every vertex $k$, the mutation at $k$ transforms $B$ into another skew-symmetric integer $n \times n$ matrix $\mu_k(B) = \overline{B} = (\overline{b}_{i,j})$. The formula for $\overline{b}_{i,j}$ is given below in (7.4). It is well-known (see Proposition 7.1 below) that the quiver $Q(\overline{B})$ can be obtained from $Q(B)$ by the following three-step procedure:

Step 1. For every incoming arrow $a : j \to k$ and every outgoing arrow $b : k \to i$, create a “composite” arrow $[ba] : j \to i$; thus, whenever $b_{i,k}, b_{k,j} > 0$, we create $b_{i,k}b_{k,j}$ new arrows from $j$ to $i$.

Step 2. Reverse all arrows at $k$; that is, replace each arrow $a : j \to k$ with $a^* : k \to j$, and $b : k \to i$ with $b^* : i \to k$.

Step 3. Remove any maximal disjoint collection of oriented 2-cycles (that can appear as a result of creating new arrows in Step 1).

In the case where $k$ is a source or a sink of $Q(B)$, the first and last steps of the above procedure are not applicable, so $Q(\overline{B})$ is obtained from $Q(B)$ by just reversing all the arrows at $k$. In this situation, J. Bernstein, I. Gelfand, and V. Ponomarev [3] introduced the reflection functor at $k$ sending representations of a quiver $Q(B)$